Bohdan Zelinka

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EDGE-COLOURINGS OF PERMUTATION GRAPHS

BOHDAN ZELINKA, Liberec

G. Chartrand and F. Harary [1] and the first author with J. B. Frechen [2] have defined the concept of permutation graph as follows.

Let G be a graph whose vertices are labelled $v_1, v_2, \ldots, v_p$ and let α be a permutation on the set $\{1, 2, \ldots, p\}$. Then by the α-permutation graph $P_\alpha(G)$ of G is meant the graph consisting of two disjoint, identically labelled copies of G, say G and G', together with p additional edges x_i, $1 \leqq i \leqq p$, where x_i joins the vertex labelled v_i in G with the vertex labelled $v_{\alpha(i)}$ in G'. To avoid a possible confusion, we often label a vertex of G' as v'_j rather than v_j.

Now in [2] the problem is asked: Let C_n denote a cycle of length n. Determine for what permutations α on $\{1, 2, \ldots, n\}$ the graph $P_\alpha(C_n)$ has the chromatic index 3, or, equivalently, 4.

The chromatic index of $P_\alpha(C_n)$ does not exceed 4, because each of the cycles C_n and C'_n in $P_\alpha(C_n)$ has the chromatic index at most 3 (the chromatic index of a cycle cannot exceed 3), therefore the edges of each of these cycles can be coloured by the colours 1, 2, 3 and each edge joining a vertex of C_n with a vertex of C'_n is then coloured by the colour 4.

This problem is closely related to a problem considered by Watkins [3].

Here we shall give some characterisation of such permutations. Of course, some more simple and practical characterisation would be needed. But it is obviously difficult to characterize some set of permutations which in general is not a group.

For any integer $n \geqq 3$ let $\mathfrak{A}(n)$ be the set of permutations α on $\{1, 2, \ldots, n\}$ for which the graph $P_\alpha(C_n)$ has the chromatic index 3. At first we shall prove a theorem for n even.

Theorem 1. *Let n be an even integer greater than three. Then $\mathfrak{A}(n)$ is the symmetric group $\mathfrak{S}_n$ of the order n, i.e. the set of all permutations on the set $\{1, 2, \ldots, n\}$.*

Proof. The graph $P_\alpha(C_n)$ consists of two cycles C_n and C'_n (both of the length n) and of edges joining vertices of C_n with vertices of C'_n and forming a linear factor of $P_\alpha(C_n)$. As n is even, the edges of C_n can be coloured by two colours 1, 2 and so can the edges of C'_n. Any edge of $P_\alpha(C_n)$ joining a vertex of C_n with a vertex of C_n' will be coloured by the colour 3. Thus we obtain a 3-colouring of the edges of $P_\alpha(C_n)$.

For n odd the situation is more complicated. We shall investigate a graph $P_\alpha(C_n)$ for some odd n and some α whose chromatic index is 3. On the cycle C_n in the graph $P_\alpha(C_n)$ we shall introduce some colouring of vertices which will be called a v-colouring. (The v-colouring need not be an admissible vertex-colouring in the usual sense, i.e. two vertices joined by an edge need not have different colours.) The v-colouring of C_n corresponding to the given admissible edge-colouring (with 3 colours) of $P_\alpha(C_n)$ will be defined as follows: Let $v_1, \ldots, v_n$ be the vertices of C_n, let $v_i v_{i+1}$ for $i = 1, \ldots, n$ be the edges of C_n (the subscripts are taken modulo n). For any i, $1 \leqq i \leqq n$, the edges $v_{i-1}v_i, v_i v_{i+1}$ have different colours in the edge-colouring, therefore their colours are some two of the colours 1, 2, 3. The vertex v_i in the v-colouring will be coloured by a colour different from these two colours and belonging to the set $\{1, 2, 3\}$. Thus the v-colouring is uniquely determined.

Lemma 1. *Any v-colouring $\mathfrak{c}$ of C_n (for n odd) corresponding to an admissible 3-colouring of edges of $P_\alpha(C_n)$ must satisfy these conditions:*
(**A**) *Not all vertices of C_n have the same colour.*
(**B**) *If for some i, $1 \leqq i \leqq n$, the vertices $v_{i+1}, \ldots, v_{i+k}$, where k is some odd integer, have all the same colour and the vertices v_i, v_{i+k+1} have colours different from this colour, then the colours of v_i and v_{i+k+1} are different from each other.*
(**C**) *If the situation is the same as sub* (**B**) *with the exception that k is even, then the colours of v_i and v_{i+k+1} are equal to each other.*

(Here the subscripts are always taken modulo n.) On the other hand, any vertex-colouring of C_n satisfying (**A**), (**B**) *and* (**C**) *is a v-colouring corresponding to some admissible 3-colouring of edges of $P_\alpha(C_n)$.*

Proof. Consider a v-colouring corresponding to some admissible 3-colouring of edges. Assume that (**A**) is not satisfied. Then all vertices of C_n have the same colour, say 1. This means that an edge of the colour 1 is incident with no vertex, therefore all edges of C_n have the colours 2 and 3, which is impossible, because n is odd. Thus (**A**) must be satisfied.

Assume that (**B**) is not satisfied, i.e. that there exists some i and k so that v_i and v_{i+k+1} are of the colour 1 and $v_{i+1}, \ldots, v_{i+k}$ are of the colour 2. (The colours were chosen without the loss of generality.) Then in the edge-colouring the edge $v_i v_{i+1}$ must have the colour 3. The edge $v_{i+1}v_{i+2}$ must have the colour different from 3, because it has a common end vertex v_{i+1} with the edge $v_i v_{i+1}$ coloured by 3, and different from 2, because it joins two vertices of the colour 2. Therefore $v_{i+1}v_{i+2}$ has the colour 1. Analogously we can prove that the colour of $v_{i+2}v_{i+3}$ is 3, the colour of $v_{i+3}v_{i+4}$ is again 1 etc. and finally the colour of $v_{i+k-1}v_{i+k}$ is 3. But then the colour of $v_{1+k}v_{i+k+1}$ must be different from 3, further different from 2, because v_{i+k} has the colour 2, and also from 1, because v_{i+k+1} has the colour 1. We have obtained a contradiction.

By a similar method we can prove that (**C**) must also be satisfied. Similarly we can also prove the inverse assertion of the lemma.

A v-coloring which satisfies (***A***), (***B***) and (**C**) will be called a w-colouring. Now we shall explain, why we have defined this concept. In the graph $P_\alpha(C_n)$ the edge joining a vertex of C_n with a vertex of C_n' must be coloured evidently by the colour of this vertex in the corresponding v-colouring (which is a w-colouring). If we introduce by the same way the v-colouring of C_n', then any pair of vertices v_i, v_j', where v_i is in C_n, v_j' is in C_n' and the edge $v_i v_j'$ belongs to $P_\alpha(C_n)$, must be coloured by the same colour in the corresponding v-colourings (which are w-colourings).

Now we shall speak about w-colourings of cyclically ordered n-tuples. Let $[y_1, \ldots, y_n]$ be some cyclically ordered n-tuple. (This means that the n-tuples $[y_1, \ldots, y_n]$, $[y_2, \ldots, y_n, y_1]$, $[y_3, \ldots, y_n, y_1, y_2], \ldots$ are considered as equal.) A w-colouring of this n-tuple is a mapping $\mathfrak{c}$ of the set $\{y_1, \ldots, y_n\}$ onto $\{1, 2, 3\}$ such that the conditions (***A***), (***B***), (**C**) are satisfied, where we write y_i instead of v_i.

The set of w-colourings of the cyclically ordered n-tuple $[1, 2, \ldots, n]$ will be denoted by $\mathfrak{W}(n)$. Now we can express a theorem.

Theorem 2. *Let n be an odd integer, $n \geqq 3$. Then the set $\mathfrak{A}(n)$ consists of all permutations α on the set $\{1, 2, \ldots, n\}$ such that for some two elements $\mathfrak{c}$, $\mathfrak{d}$ of $\mathfrak{W}(n)$ the equality $\mathfrak{c} = \mathfrak{d}\alpha$ holds.*

Remark. Here $\mathfrak{d}\alpha$ means (as usual) the superposition (product) of the mappings α and $\mathfrak{d}$.

Proof. Consider the graph $P_\alpha(C_n)$ for $\alpha \in \mathfrak{A}(n)$. Let $\mathfrak{c}$ be the w-colouring on C_n and $\mathfrak{d}$ the w-colouring of C_n' corresponding to some admissible edge-colouring of $P_\alpha(C_n)$ by three colours. The vertices v_i on C_n can be considered simply as numbers i; so can the vertices v_i' on C_n'. Thus the w-colourings $\mathfrak{c}$ and $\mathfrak{d}$ are considered as w-colourings of the cyclically ordered n-tuple $[1, 2, \ldots, n]$. In the graph $P_\alpha(C_n)$ any pair v_i, $v_{\alpha(i)}$ is joined by an edge and thus (according to the above considerations) the colours of these vertices in corresponding w-colourings are equal; this means $\mathfrak{c}(i) = \mathfrak{d}\alpha(i)$. As this holds for an arbitrary i, we have $\mathfrak{c} = \mathfrak{d}\alpha$. It is evident that also for every α for which $\mathfrak{c} = \mathfrak{d}\alpha$ holds (for some $\mathfrak{c}$ and $\mathfrak{d}$ from $\mathfrak{W}(n)$) we can colour the edges of $P_\alpha(C_n)$ by three colours.

Thus we have characterized the elements of $\mathfrak{A}(n)$ with the help of the set $\mathfrak{W}(n)$. From the equality $\mathfrak{c} = \mathfrak{d}\alpha$ the mapping α cannot be determined uniquely, because $\mathfrak{c}$ and $\mathfrak{d}$ are many-to-one mappings. But if we know the set $\mathfrak{W}(n)$, we can construct also $\mathfrak{A}(n)$.

Now we shall study $\mathfrak{W}(n)$ for odd n. At first we shall prove a theorem.

Theorem 3. *In a w-colouring of the n-tuple* $[1, 2, \ldots, n]$ *for odd n all three colours occur.*

Proof. According to (**A**) not all numbers $1, \ldots, n$ have the same colour. Assume that in some w-colouring $\mathfrak{c}$ only the colours 1 and 2 occur. For some i and k_1 let $\mathfrak{c}(i) = 1, \mathfrak{c}(i+1) = \mathfrak{c}(i+2) = \ldots = \mathfrak{c}(i+k_1) = 2, \mathfrak{c}(i+k_1+1) = 1$. Then, according to (**B**), the number k_1 must be even. Now if for some k_2 we have $\mathfrak{c}(i+k_1+1) = \mathfrak{c}(i+k_1+2) = \ldots = \mathfrak{c}(i+k_1+k_2) = 1, \mathfrak{c}(i+k_1+k_2+1) = 2$, this k_2 must also be even. Thus we continue further and at the end we must have a sequence of even numbers $k_1, \ldots, k_n$ such that $\sum_{i=1}^{m} k_i = n$. But n is an odd number, which is a contradiction.

Now let $\mathfrak{G}(n)$ be the group generated by all cyclic permutations of the n-tuple $[1, 2, \ldots, n]$ and by the mirror permutation μ defined so that $\mu(i) = n + 1 - i$ for $i = 1, \ldots, n$.

Lemma 2. *If* $\mathfrak{c} \in \mathfrak{W}(n)$, $\beta \in \mathfrak{G}(n)$, *then* $\mathfrak{c}\beta \in \mathfrak{W}(n)$.

Proof is simple. At any cyclic permutation and at the mirror permutation the conditions (**A**), (**B**), (**C**) are evidently preserved and thus they are preserved also at any permutation of $\mathfrak{G}(n)$.

From this lemma a theorem follows.

Theorem 4. *Let $n \geqq 3$ be an odd integer. If* $\alpha \in \mathfrak{A}(n)$, $\beta \in \mathfrak{G}(n)$, *then* $\alpha\beta \in \mathfrak{A}(n)$.

Proof. As $\alpha \in \mathfrak{A}(n)$, there exist elements $\mathfrak{c}$, $\mathfrak{d}$ of $\mathfrak{W}(n)$ so that $\mathfrak{c} = \mathfrak{d}\alpha$. According to Lemma 2 we have $\mathfrak{c}\beta \in \mathfrak{W}(n)$. For the element $\alpha\beta$ we have $\mathfrak{c}\beta = \mathfrak{d}\alpha\beta$, where $\mathfrak{c}\beta \in \mathfrak{W}(n)$, $\mathfrak{d} \in \mathfrak{W}(n)$, therefore $\alpha\beta \in \mathfrak{A}(n)$.

Now we shall determine $\mathfrak{W}(n)$ and $\mathfrak{A}(n)$ for some small n.

For $n = 3$ the set $\mathfrak{W}(3)$ consists of the triple $[1, 2, 3]$ and the triples obtained from this by the permutations of $\mathfrak{G}(3)$ and by interchanging colours. As $\mathfrak{G}(3)$ is the symmetrical group $\mathfrak{S}_3$ of the order 3, any triple consisting of (all) the numbers 1, 2, 3 is a w-colouring and $\mathfrak{A}(3)$ is the symmetrical group of the order 3.

For $n = 5$ the set $\mathfrak{W}(5)$ consists of the quintuple $[1, 2, 3, 3, 3]$ and the quintuples obtained from this by the permutations of $\mathfrak{G}(5)$ and by interchanging colours. The set $\mathfrak{A}(5)$ consists of the permutations which transform some triple of consecutive (in the cyclic order) elements again onto such a triple (of the elements coloured by the same colour). Evidently the remaining terms are two consecutive terms, therefore the set $\mathfrak{A}(5)$ consists exactly of all permutations which transform some pair of consecutive terms again onto a pair of consecutive terms. In other words, the complement of $\mathfrak{A}(5)$ in $\mathfrak{S}_5$ is the set of all isomorphic mappings of the cycle C_5 onto its complement (which is also a cycle).

For the case $n = 7$ we shall determine only $\mathfrak{W}(7)$. It consists of [1, 2, 3, 3, 3, 3, 3] and [1, 2, 3, 1, 1, 3, 3] and all septuples obtained from these by the permutations of $\mathfrak{G}(7)$ and by interchanging colours.

Finally we shall prove again some general theorems.

Theorem 5. *For any odd integer $n \geqq 3$ any permutation on $\{1, 2, \ldots, n\}$ not belonging to $\mathfrak{A}(n)$ maps C_n isomorphically into its complement.*

Remark. For $n = 5$ we had an onto-mapping, because C_5 is self-complementary. In general we have an into-mapping, i.e. a mapping onto some subgraph of the complement of C_n.

Proof. Assume that there exists $\alpha \notin \mathfrak{A}(n)$ which does not map C_n isomorphically into its complement. This means that at least one edge of C_n, i.e. a pair of consecutive (in the cyclical order) numbers of $\{1, \ldots, n\}$ is mapped by α again onto an edge. Then there exist some i, j so that $\alpha(i) = j$, $\alpha(i + 1) = j + 1$ or $\alpha(i + 1) = j - 1$. Let $\mathfrak{c}$ be the w-colouring such that $\mathfrak{c}(i) = 1$, $\mathfrak{c}(i + 1) = 2$, $\mathfrak{c}(k) = 3$ for other k. Let $\mathfrak{d}$ be the w-colouring such that $\mathfrak{d}(j) = 1$, $\mathfrak{d}(j + 1) = 2$ if $\alpha(i + 1) = j + 1$, and $\mathfrak{d}(j + 1) = 3$ if $\alpha(i + 1) = j - 1$, $\mathfrak{d}(j - 1) = 2$ if $\alpha(i + 1) = j - 1$ and $\mathfrak{d}(j - 1) = 3$ if $\alpha(i + 1) = j + 1$, $\mathfrak{d}(k) = 3$ for other k. The mappings $\mathfrak{c}$, $\mathfrak{d}$ are in $\mathfrak{W}(n)$, because they satisfy (**A**), (**B**), (**C**). And now $\mathfrak{c} = \mathfrak{d}\alpha$, therefore $\alpha \in \mathfrak{A}(n)$, which is a contradiction.

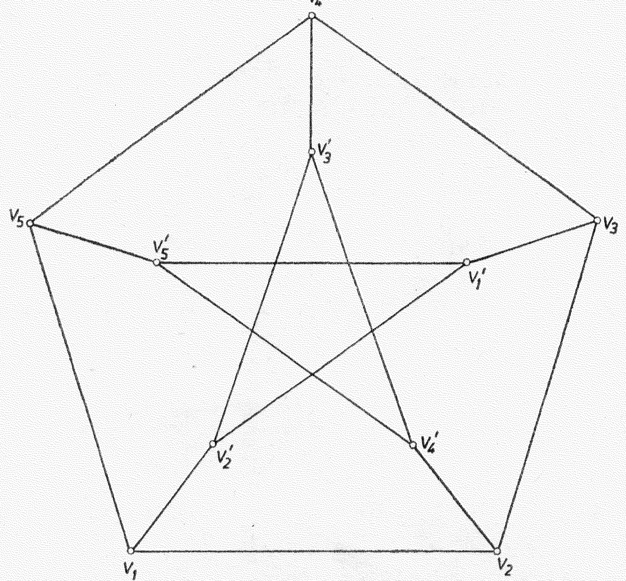

Fig. 1

Now a permutation α of the set $\{1, \ldots, n\}$ in which for some i and j, $i \neq$ we have $\alpha(i) = j$, $\alpha(j) = i$, $\alpha(k) = k$ for all k different from i and j, will be called an elementary exchange.

Theorem 6. *Let α be an elementary exchange on the set $\{1, \ldots, n\}$, where n is odd, $n \geqq 3$. Then $\alpha \in \mathfrak{A}(n)$.*

Proof. For $n = 3$ this was proved above. Let $n \geqq 5$. Let $\alpha(i) = j$, $\alpha(j) = i$, $\alpha(k) = k$ for another k. There exists some l such that both l and $l + 1$ (taken modulo n) are different from both i and j. Let $\mathfrak{c}$ be such a w-colouring that $\mathfrak{c}(l) = 1$, $\mathfrak{c}(l + 1) = 2$, $\mathfrak{c}(k) = 3$ for all other k. This is evidently a w-colouring and $\mathfrak{c}\alpha$ is also a w-colouring, which implies $\alpha \in \mathfrak{A}(n)$.

Theorem 7. *For every odd integer $n \geqq 3$ the set $\mathfrak{A}(n)$ is either equal to the symmetric group $\mathfrak{S}_n$ of the order n, or is not a group.*

Proof. The elementary exchanges form a full system of generators of the symmetric group. As they all (according to Theorem 6) belong to $\mathfrak{A}(n)$, the theorem is proved.

For $n = 3$ the set $\mathfrak{A}(3)$ is a symmetric group $\mathfrak{S}_3$ of the order 3, as proved above. For $n = 5$ the permutation α for which $\alpha(i) \equiv 2i \pmod 5$ for $i = 1, \ldots, 5$ is not in $\mathfrak{A}(5)$. For this α the graph $P_\alpha(C_5)$ is the well-known Petersen graph (Fig. 1).

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