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AN ISEOMORPHISM THEOREM FOR POSITIVE COMMUTATIVE SEMIGROUPS ON THE PLANE[1]

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Abstract. A positive semigroup is a semigroup which has a copy of the nonnegative real numbers embedded as a closed subset in such a way that 0 is a zero and 1 is an identity. A positive Clifford semigroup is a positive semigroup which is the union of groups. The unpublished question of whether or not two positive commutative semigroups on E^2 whose semilattices of idempotent elements are isomorphic must be iseomorphic has been posed by J. G. Horne, Jr. In this work the question is answered in the negative with a counterexample, and necessary and sufficient conditions are given in order that two positive commutative Clifford semigroups on E^2 be iseomorphic.

1. Introduction

A topological semigroup is a Hausdorff space together with a continuous associative multiplication. The author has defined a *positive semigroup* to be a topological semigroup containing a subsemigroup N iseomorphic to the multiplicative semigroup of nonnegative real numbers, embedded as a closed subset of E^2 so that 1 is an identity and 0 is a zero [3]. Such semigroups which meet the additional requirement of being the union of groups are called *positive Clifford semigroups* [4]. The unpublished question of whether or not two positive commutative semigroups on E^2 whose semilattices of idempotent elements are isomorphic must be iseomorphic has been posed by J. G. Horne, Jr. The counter example which we will give presently answers the question in the negative. Following the example we will give necessary and sufficient conditions in order that two positive commutative Clifford semigroups on E^2 be iseomorphic.

2. Preliminaries

The closure of a subset A of a topological space is denoted $\bar{A}$. The set-theoretic difference of two sets A and B is denoted $A \backslash B$. An *iseomorphism*

[1]) This paper contains part of a doctoral dissertation written under the direction of Professor D. R. Brown at the University of Tennessee.

between two topological semigroups is a function which is both an algebraic isomorphism and a homeomorphism. The inverse of an element s is denoted s^{-1}. The set $H(1)$ denotes the set of elements with inverses with respect to the identity element 1. In general $H(e)$ denotes the maximal group having e as identity [1, p. 22]. Let G denote the component of the identity of $H(1)$. Throughout this work E^2 will denote the Euclidean plane. We will use the terminology *two dimensional* to mean having an interior relative to E^2, and *one dimensional* to mean nontrivial but having no interior relative to E^2. Unless otherwise indicated, R will denote a semigroup iseomorphic to the multiplicative semigroup of real numbers. The set of all positive members of R is denoted P, and the set of all negative members by $-P$. The set of all nonnegative members of R, i.e. $P \cup \{0\}$ is denoted by N. The null set is denoted by $\square$. For additional terminology the reader is referred to [3] and [5].

3. A Counter Example

Henceforth S will denote a positive Clifford semigroup on E^2. We intend so imply that a fixed iseomorphic copy of the nonnegative real numbers has been chosen.

Example. Let us consider three copies of $N \times N$. Let these copies be denoted $J \times J$, $M \times M$, and $Q \times Q$. Let us define a relation R on $T = [(J \times J) \cup (M \times M) \cup (Q \times Q)]$ by first requiring that $\Delta \subset R$. In addition, let us define $[(a, b)_j, (c, d)_m] \in R$ if and only if $[(c, d)_m, (a, b)_j] \in R$ if and only if $a = 0 = d$ and $b = c$, where $(a, b)_j \in (J \times J)$ and $(c, d)_m \in (M \times M)$. Finally, let us define $[(c, d)_m, (x, y)_q] \in R$ if and only if $[(x, y)_q, (c, d)_m] \in R$ if and only if $c = 0 = y$ and $d = x$, where $(x, y)_q \in (Q \times Q)$. Let us now define $(a, b)_j \cdot (c, d)_m = (c, d)_m \cdot (a, b)_j = (0, bc)_j = (bc, 0)_m$, $(a, b)_j \cdot (x, y)_q = (x, y)_q \cdot (a, b)_j = (0, 0)_j$, and $(x, y)_q \cdot (c, d)_m = (c, d)_m \cdot (x, y)_q = (0, xd)_m = (xd, 0)_q$. This multiplication is easily checked to be associative. The continuity of the multiplication follows from the continuity of real number multiplication. Let us now take another copy of $N \times N$, which we will denote simply as $N \times N$, and let us define multiplication between T and $N \times N$ in the following manner. Let $(c, d)_n \cdot (x, y)_j = (x, y)_j \cdot (c, d)_n = (cx, cdy)_j$, $(c, d)_n \cdot (z, w)_q = (z, w)_q \cdot (c, d)_n = (c^2dz, dw)_q$, and $(c, d)_n \cdot (a, b)_m = (a, b)_m \cdot (c, d)_n = (cda, c^2db)_m$. Let multiplication be coordinatewise within $J \times J$, $M \times M, Q \times Q$, and $N \times N$. It is not difficult to check that this multiplication is continuous and associative. Let us define a relation R^* on $[T \cup (N \times N)]$ by requiring that $\Delta \subset R^*$, $[(a, b)_j, (x, y)_n] \in R^*$ if and only if $[(x, y)_n, (a, b)_j] \in R^*$ if and only if $a = x$ and $b = 0 = y$, and $[(a, b)_q, (x, y)_n] \in R^*$ if and only if $[(x, y)_n, (a, b)_q] \in R^*$ if and only if $a = 0 = x$ and $b = y$. Then, it is clear that R^* is an equivalence relation, and it can be checked that R^* i] a closed congruence. It is also not difficult to show that $[T \cup (N \times N)]$ is a Hausdorff space [3, p. 33]. Thus, we have constructed an example of a positive commutative semigroup on E^2.

Let us now construct a second example in the same fashion, except that we will define the multiplication between T and $N \times N$ a little differently. Let $(c, d)_n \cdot (x, y)_j = (x, y)_j \cdot (c, d)_n = (cx, cdy)_j$, $(c, d)_n \cdot (z, w)_q = (z, w)_q \cdot \cdot (c, d)_n = (cdz, dw)_q$, and $(c, d)_n \cdot (a, b)_m = (a, b)_m \cdot (c, d)_n = (cda, cdb)_m$. Again we have an example of a positive commutative semigroup on E^2. It is apparent that these two semigroups have isomorphic semilattices of idempotent elements. However, the two semigroups are not iseomorphic. For, let us consider the translation of $H(1)$ (the interior of $N \times N$) in the first example by $(1, 0)_q$. Now, $(1, 0)_q \cdot (c, d)_n = (1, 0)_q$ if and only if $c^2 d = 1$. So. the kernel of this translation homomorphism is $\{(x^2, 1/x^2)\}$ in $H(1)$. Let us next consider the translation of $H(1)$ in the second example by $(1, 0)_q$. We have $(1, 0)_q \cdot \cdot (c, d)_n = (1, 0)_q$ if and only if $cd = 1$. Thus, the kernel of this translation homomorphism is $\{(x, 1/x)\}$ in $H(1)$. Hence, the kernels are different, and consequently the two semigroups cannot be iseomorphic.

4. An Iseomorphism Theorem

Let us now consider a lemma which will be used in one case of the proof of the main theorem. The main theorem will furnish necessary and sufficient conditions under which two positive commutative Clifford semigroups on E^2 are iseomorphic.

Lemma. *If β is an iseomorphism such that β takes the one parameter subgroup $\{(x, x^r) : x \in P, r \in R, r$ fixed $\neq 0\}$ under coordinatewise multiplication onto the one parameter subgroup $\{(y, y^s) : y \in P, s \in R, s$ fixed $\neq 0\}$ under coordinatewise multiplication, then β can be extended on E^2 in such a way that β takes $\{(x, 1)\}$ onto $\{(y, 1)\}$ and $\{(1, x)\}$ onto $\{(1, y)\}$.*

Proof. Let β' be any iseomorphism such that β' takes $\{(x, 1)\}$ onto $\{(y, 1)\}$. If $(a, b) \in E^2$, we can write $(a, b) = (x_1, x_1^r) \cdot (x_2, 1)$ for a unique x_1 and x_2, since $\{(x, x^r)\}$ and $\{(x, 1)\}$ form a basis for the space. Let us define $\hat{\beta}[(a, b)] = = \beta[(x_1, x_1^r)] \cdot \beta'[(x_2, 1)]$. It is easily checked that $\hat{\beta}$ is an iseomorphism of E^2 onto E^2 which is an extension of β. Now, β and β' are determined by their action on any point. Suppose that $\beta[(x_1, x_1^r)] = (y_1, y_1^s)$ and $\beta'[(x_2, 1)] = = (y_2, 1)$. By choosing y_2 properly we can determine β' such that $\hat{\beta}$ is the required extension. We must be able to pick y_2 so that $\hat{\beta}[(1, x_3)] = (1, y_3)$. Now, $\hat{\beta}[(1, x_3)] = \hat{\beta}[(x_1, x_1^r)^p \cdot (x_2, 1)^q] = (y_1^p, y_1^{sp}) \cdot (y_2^q, 1) = (y_1^p y_2^q, y_1^{sp})$.

So all we need to do is to pick y_2 so that $y_1^p y_2^q = 1$, which can be accomplished by selecting $y_2 = y_1^{-p/q}$.

In the main theorem which we are now ready to state, we will consider two positive commutative Clifford semigroups on E^2 which we will call S and S'. Accordingly, we will denote by G and G' the identity components of $H(1)$ and $H(1')$, respectively. Also, if Ψ is an isomorphism from the semilattice of idempotent elements of S onto the semilattice of idempotent elements of S', we will denote by e' the element $\Psi(e)$. Furthermore, we will denote by K_e the kernel of the translation of G by e and by $K_{e'}$ the kernel of the translation of G' by e'.

Theorem. *Let S and S' be positive commutative Clifford semigroups on E^2. Suppose that Ψ is an isomorphism from the semilattice of idempotent elements of S onto the semilattice of idempotent elements of S'. Furthermore, suppose that there is an iseomorphism Φ from $\bar{G}$ to $\bar{G}'$ such that Φ agrees with Ψ on the idempotent elements of G, and such that $\Phi(K_e) = K_{e'}$, for each e such that e is the idempotent element of a one dimensional group in S distinct from the bounding ray of G. Then, S is iseomorphic to S'.*

Proof. The proof of the theorem will be initiated with six preliminary cases, with the final required iseomorphism being exhibited in terms of the iseomorphisms achieved in these cases. In the first two cases we will consider $\bar{G} \cup \bar{C}$, where C is the identity component of a two dimensional group which shares a bounding ray with G. The first two diagrams in Figure 1 might be helpful in visualizing these two cases. In these two cases, as well as in the remaining four cases, $\bar{G} \cup \bar{C}$ is a subsemigroup of S. For, we know that C and G are groups. Furthermore, if $c \in C$, cG is the continuous homomorphic image of a group, and is hence a group. Since cG meets C in c, $cG \subset C$, and by continuity $c\bar{G} \subset \bar{C}$. It is not difficult to see that there are only two possible cases when C and G share a bounding ray. For, K_g must be a one dimensional group such that $0 \notin \bar{K}_g$. Otherwise, $0 \cdot g = g$ by continuity, which is a contradiction. Moreover, $e \notin \bar{K}_g$, else $eg = g$ by continuity, which is also a contradiction, since C is iseomorphic to $N \times N$.

In the final four cases we will consider $\bar{G} \cup \bar{C}$, where $\bar{G} \cap \bar{C} = \{0\}$. The last four diagrams in Figure 1 should be helpful here. If we recall that $\bar{C} \,.\, \bar{G} \subset \bar{C}$, then arguments like those which will be given in the proof of Case (iii) to determine the product of idempotent elements from $\bar{C}$ and $\bar{G}$ will show that these cases will in general be exhaustive.

Case (i). Let us first consider the union of the closure of the identity component of $H(1)$ with the closure of the identity component of another two dimensional group $H(j)$ which shares a bounding ray with the identity component of $H(1)$. Let us denote the identity component of $H(j)$ by C. Let e denote the idempotent element on the bounding ray shared by G and C. Let g denote the idempotent element on the other bounding ray of G. In this case, let us assume that K_g is a closed subgroup of G. Let Φ be the iseomorphism from $\bar{G}$ onto $\bar{G}'$ such that $\Phi(K_g) = K_g'$, where $g' = \Phi(g) = \Psi(g)$. Now jG is the continuous homomorphic image of the group G, and is hence itself a group which contains j. Now, $je = e$, and hence $j \,.\, (Ne) = Ne$, where N is the non-negative real numbers. Since jG is a group containing j and whose closure contains Ne, the bounding ray between C and D, jG must be two dimensional.

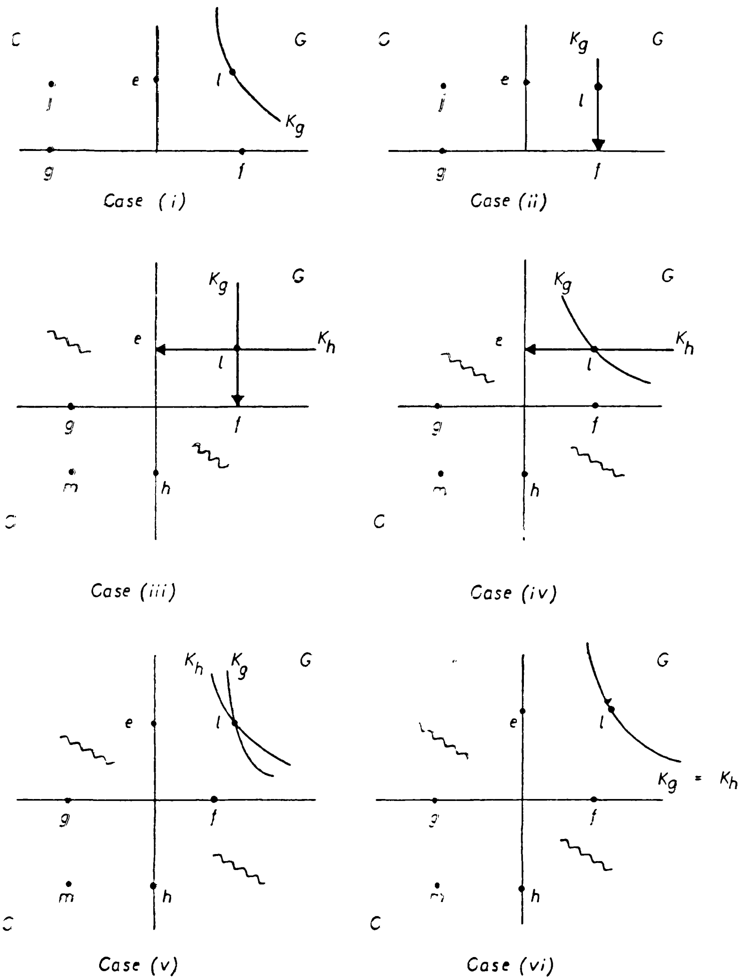

Fig. 1. Diagrams for the cases in the theorem.

Thus jG must be C. Similarly, $gG = D$, where D is the identity component of $H(g)$. The translation of G by j is also one-to-one, since it is a linear transformation of a two dimensional vector space onto a two dimensional vector space [1, p. 208]. Let us extend Φ to C in the following manner. Let W be the ray shared by $\bar{C}$ and $\bar{G}$. Thus, $W = Ne = Pe \cup \{0\}$. If $y \in (C \cup W)$, then $y = jx$ for a unique x in $(G \cup W)$, since $jw = w$ for all $w \in W$. Let us define Φ_j from $C \cup W$ into S' by $\Phi_j(y) = \Psi(j) \cdot \Phi(x)$. Since Φ is defined on $\bar{C}$, it is defined in particular on W so that the map Φ_j is well-defined and is continuous. Now, if $W' = \Phi(W)$ and C' is the identity component of $H(\Psi(j))$, then since $\Phi|_w = \Phi_j|_w$, we have $\Phi_j(C \cup W) \subset C' \cup W'$. Let $w' \in W'$ so that $w' = \Phi(w)$, and let $\{c_n\}$ be a sequence in C converging to w. Then $\{\Phi_j(c_n)\}$ is a sequence in C' converging to $\Phi_j(w) = \Phi(w) = w'$. Hence, $W' \subset \bar{C}'$. Thus, C' must be one of the two dimensional groups sharing a bounding ray with G', in which case the j' translate of G' is one-to-one from G' onto C'. Let us show that Φ_j is onto. Let $y' \in C'$. Then, there is a unique x' in G' such that $y' = j'x' = j'\Phi(x)$, where x is an element of G. But, $j'\Phi(x) = \Phi_j(jx)$, where $jx \in C$. Now, let us show that Φ_j is one-to-one. Suppose that $\Phi_j(y_1) = \Phi_j(y_2)$, where $y_1 = jx_1$, $y_2 = jx_2$, and x_1, x_2 are elements of G. Then $\Phi_j(jx_1) = \Phi_j(jx_2)$, whence $j'\Phi(x_1) = j'\Phi(x_2)$. Since the translation of G' by j' is one-to-one, we have $\Phi(x_1) = \Phi(x_2)$, whence $x_1 = x_2$, and $jx_1 = jx_2$.

We can also extend Φ to D in the following manner. Since $gG = D$, for an element z in D there is an x in G such that $gx = z$. Let us define $\Phi_g(z) = g'\Phi(x)$. We must show that Φ_g is well-defined. Suppose $z = gx_1 = gx_2$, where $x_1, x_2 \in G$. Then, $\Phi_g(z) = g'\Phi(x_1)$, and $\Phi_g(z) = g'\Phi(x_2)$. We must show that $g'\Phi(x_1) = g'\Phi(x_2)$. Now, $gx_1 = gx_2$ implies that $g(x_1x_2^{-1}) = g$, whence $(x_1x_2^{-1}) \in K_g$. Since $\Phi(K_g) = K_{g'}$, we have that $\Phi(x_1x_2^{-1}) = [\Phi(x_1)] \cdot [\Phi(x_2)]^{-1} \in K_{g'}$, so that $g'[\Phi(x_1)] \cdot [\Phi(x_2)]^{-1} = g'$, and $g'\Phi(x_1) = g'\Phi(x_2)$. It is not difficult to show that Φ_g is one-to-one and onto. The proof that Φ_g is continuous while lengthy can be done in a straight forward manner with sequences.

Let us denote by $\hat{\Phi}$ the extension of Φ to $\bar{C}$. Let us show that $\hat{\Phi}$ preserves multiplication. Let $x, y \in C$, such that $x = ja$, $y = jb$, where $a, b \in G$. Then $\hat{\Phi}(xy) = \Phi_j(xy) = \Phi_j(jab) = j'\Phi(ab) = [j'\Phi(a)] \cdot [j'\Phi(b)] = [\Phi_j(ja)] \cdot [\Phi_j(jb)] = [\hat{\Phi}(x)] \cdot [\hat{\Phi}(y)]$. The other cases to be checked are similar to this one.

Case (ii). This case will differ from the first one only that here we will assume that f is in the closure of K_g. Despite this difference, we can handle this case in exactly the same manner as Case (i).

Case (iii). In this case we will consider the union of $\bar{G}$ and $\bar{C}$, where C is the identity component of a two dimensional group $H(m)$ with the property that $\bar{C} \cap \bar{G} = \{0\}$. Let e and f be the idempotent elements on the bounding rays of G, and let g and h be the idempotent elements on the bounding rays of C such that the points $\Phi(e)$, $\Phi(f)$, $\Phi(h)$, and $\Phi(g)$ on the decomposition circle appear clockwise in the order listed. More particularly, let us assume in this case that e is in the closure of K_h and that f is in the closure of K_g. Again, we know that mG is a group contained in C. So, by continuity of multiplication, $me = g$, $me = 0$, $me = m$, or $me = h$. Also, $he = h$ by continuity of multiplication, and similarly $ge = m$, $ge = h$, or $ge = 0$. But, $ge = m$ implies that $geh = mh = h$, which is a contradiction, since $gh = 0$. Also, $ge = h$ implies that $gge = ge = gh$ which implies that $gh = h$, which is a contradiction, since $gh = 0$. So, we conclude that $ge = 0$. Then, if $me = m$, $gem = gm = 0$, which is a contradiction, since $gm = g$. If $me = g$, then $meh = gh$, $mh = h = gh = 0$, which is again a contradiction. If $me = 0$, then $meh = mh = h = 0$, which is again a contradiction. So, we conclude that $me = h$. It also follows by similar eliminations that $gf = mf = hf = 0$. Now, let $\{y_n\}$ be a sequence in K_h such that $\{y_n\}$ converges to e. Then, $\{my_n\}$ is a sequence in C such that $\{my_n\}$ converges to $me = h$. Thus, not all of the elements of the sequence $\{my_n\}$ are on the ray Pm. So, mG is not a one dimensional group in C, but is rather a two dimensional group in C, from which we conclude that $mG = C$. Now, this case can be handled in the same manner as Case (i).

Case (iv). This case will differ from the previous one only in that here we will assume that K_g is a closed subgroup of G. However, just as in the previous case, $mG = C$, since no use was made of the fact that $f \in \bar{K}_g$. So this case can also be handled in the same manner as Case (i).

Case (v). This case will differ from Case (iv) only in that we will assume that K_g and K_h are distinct closed subgroups of G. In this case $ge = h$, $ge = m$, or $ge = 0$. But, $ge = h$ implies that $geh = hh = h$, which is a contradiction, since $gh = 0$. Also, $ge = m$ implies that $geh = mh = h$, which is a contradiction, since $gh = 0$. So, we conclude that $ge = 0$. Now, suppose that $me = g$. Then $meg = gg = g$, which is a contradiction, since $eg = 0$. It is clear that mK_g and mK_h are contained in S_g and S_h (in C) respectively. Not both of mK_g and mK_h can be the point m. This is true since K_g and K_h form a basis for G, whence mG would be the point m, and by continuity of multiplication $m \cdot 0 = m$, which is a contradiction. So, we may assume without loss of generality that $mK_g = S_g$. But, $mS_e \subset mK_g$, since $me \neq g$. So, mG is not one dimensional, whence $mG = C$. Now this case can be handled in the same manner as Case (i).

Case (vi). This case will differ from the previous one only in that here we will assume that $K_g = K_h$. In this case $me = m$, $me = g$, $me = h$, or $me = 0$. But $me = m$ implies that $gme = gm$, and $ge = e$, which is a contradiction, since $e \notin \bar{K}_g$. Likewise, $me = g$ implies that $gme = gg$, and $ge = g$, and we have the same contradiction. Let us now suppose that $me = h$. Then, $meh = hh = h$, $he = h$, which is a contradiction since $e \notin \bar{K}_h$. So, we conclude that $me = 0$. Similarly, we have that $mf = 0$. Hence, mS_e and mS_f must both be the ray Pm. Since S_e and S_f are a basis for G, we have that mG is the one dimensional group Pm.

We can extend Φ to $Pm = mG$ in the following manner. Let $y \in mG$. Then, $y = mx$, for some $x \in G$. Let us define $\Phi_m(y) = m'\Phi(x)$. First, we must show that Φ_m is well-defined. Suppose that $y = mx_1 = mx_2$, where $x_1, x_2 \in G$. So, $m(x_1x_2^{-1}) = m$, and $(x_1x_2^{-1}) \in K_m$. But $K_m \subset K_g$, for if $z \in G$ and $zm = m$, then $zg = zmg = mg = g$. So, if $K_m \neq \{1\}$, $K_m = K_g$, since both are one dimensional groups. Thus, $\Phi(x_1x_2^{-1}) = \Phi(x_1)[\Phi(x_2)]^{-1} \in K_{m'}$, whence $m' [\Phi(x_1)] \cdot [\Phi(x_2)]^{-1} = m'$, and $m'\Phi(x_1) = m'\Phi(x_2) = \Phi_m(y)$. Let us next show that Φ_m is one-to-one. Suppose that $\Phi_m(mx_1) = \Phi_m(mx_2)$, where $x_1, x_2 \in G$. Then, $m'\Phi(x_1) = m'\Phi(x_2)$, and $m'\Phi(x_1x_2^{-1}) = m'$. But, since $\Phi(x_1x_2^{-1}) \in K_{m'}$, $\Phi(K_m) = K_{m'}$, and Φ is one-to-one, we have that $x_1x_2^{-1} \in K_m$, or $m(x_1x_2^{-1}) = m$, and $mx_1 = mx_2$. The proof that Φ is continuous is similar to that in the previous cases.

By use of the Lemma we can extend Φ_m to an iseomorphism Φ^* from C to C' so that $\Phi^*(S_g) = S_g'$ and $\Phi^*(S_h) = S_{h'}$. Then, we can extend Φ^* to an iseomorphism of C onto C' by defining, as before the iseomorphism Φ_g from the identity component of $H(g)$ onto the identity component of $H(g')$, and by defining Φ_h from the identity component of $H(h)$ onto the identity component of $H(h')$. We must now show that this extension is continuous. Let $\{y_n\}$ be a sequence in C such that $\{y_n\}$ converges to g. Since the one dimensional group mG and the one dimensional group S_g form a basis for C, $\{y_n\} = \{mx_na_n\}$, where each $x_n \in G$ and each $a_n \in S_g$. So, $\{mx_na_n\}$ converges to g, $\{gmx_na_n\} = \{gx_n\}$ converges to $gg = g$, and by the continuity of Φ_g, $\{g'\Phi(x_n)\}$ converges to g'. Since Φ^* is an iseomorphism, $\{\Phi^*(y_n)\} = \{\Phi^*(mx_na_n)\} = \{\Phi^*(mx_n)\Phi^*(a_n)\} = \{m'\Phi(x_n)\Phi^*(a_n)\}$ which clusters to t, so that $\{g'm'\Phi(x_n)\Phi^*(a_n)\}$ $\{g'\Phi(x_n)\}$ converges to $g't$. So, $g't = g'$, whence $t = g'$ or $t \in S_{g'}$ in C'. The latter assumption leads to a contradiction, since Φ^* is an iseomorphism and $\{y_n\}$ does not cluster to a point in C. Now, in more generality, let $\{y_n\}$ be a sequence in C such that $\{y_n\}$ converges to p, where p is an element of the identity component of $H(g)$. Then, $\{p^{-1}y_n\} = \{gp^{-1}y_n\} = \{gp^{-1}mx_na_n\} = \{gp^{-1}x_n\}$ converges to g. But, $p^{-1} = gx_m$, for some $x_m \in G$, and $p = gx_m^{-1}$. So, $\{gx_nx_m\}$ converges to g, whence $\{g'\Phi(x_nx_m)\} = \{g'\Phi(x_n)\Phi(x_m)\}$ converges to g', and $\{g'\Phi(x_m)\}$ converges to $g'\Phi(x_m^{-1})$. Thus, $\{\Phi_g(gx_n)\} = \{g'\Phi(x_n)\}$ converges to $g'\Phi(x_m^{-1}) = \Phi_g(p)$. We must show that $\{\Phi^*(y_n)\}$ converges to $\Phi_g(p)$. But, $\{\Phi^*(y_n)\}$ clusters to t, whence $\{g'\Phi^*(y_n)\} = \{g'm'\Phi(x_n)\Phi^*(a_n)\} = \{g'\Phi(x_n)\}$ clusters to $g't$. Thus $g't = \Phi_g(p)$ and $g't\Phi_g(p^{-1}) = g'$. But, $t\Phi_g(p^{-1})$ is an element of the identity component of $H(g')$, so that $t\Phi_g(p^{-1}) = g'$, and $t = g'\Phi_g(p) = \Phi_g(p)$. In exactly the same manner, it can be shown that if $\{y_n\}$ is a sequence in C such that $\{y_n\}$ converges to p, an element of the identity component of $H(h)$, then $\{\Phi^*(y_n)\}$ converges to $\Phi_h(p)$.

We have now shown how to extend the iseomorphism from $\bar{G}$ onto $\bar{G}'$ to the closures of identity components of other two dimensional groups in S. The iseomorphism can be extended to non-identity components in the following manner. Suppose that $H(j)$ is a disconnected maximal group in S. Let D_o be the identity component of $H(j)$ and let D_j be another component. We now use the fact that there is a unique element x such that $x^2 = 1$ and $x\bar{D}_o = \bar{D}_j$. If $z \in \bar{D}_j$, $z = xa$ for a unique element a in $\bar{D}_o$. Let us define $\Phi(z) = y\Phi(a)$. where y is a unique element such that $y^2 = 1$ and $y\bar{D}_o' = \bar{D}_j'$. Suppose that $z = xa$, $w = xb$, where $a, b \in \bar{D}_o$. If $\Phi(z) = \Phi(w)$, $y\Phi(a) = y\Phi(b)$. $y^2\Phi(a) = y^2\Phi(b)$. $\Phi(a) = \Phi(b)$, $a = b$, and $z = w$. So, this extension is one-to-one. The extension is clearly onto, and since multiplication is continuous, it follows that the extension and its inverse are continuous. Also, $\Phi(z)\Phi(w) = [y\Phi(a)] \cdot$

$$\cdot [y\Phi(b)] = \Phi(a)(yy)\Phi(b) = \Phi(a)\Phi(b) = \Phi(ab) = \Phi(axxb) = \Phi(xaxb) = \Phi(zw).$$

Let us now show how to extend our iseomorphism of $\bar{G}$ onto $\bar{G}'$ to a sector of one dimensional groups in S. We will first show how to do this in the case of a sector of connected one dimensional groups, and then the extension can be carried out when the one dimensional groups have more than one component by translating the identity component sector of such groups by a square root of 1 in a non-identity component sector in the same way as just done above. Now, let $C = \bar{C}\backslash\{0\}$ be a maximal sector of connected one dimensional groups. Just as above, we can show that $\bar{G} \cup \bar{C}$ is a subsemigroup of S. Let us denote the set of non-zero idempotent elements of $\bar{C}$ by F. Let P be the copy of the positive real numbers embedded in G, and let $N = P \cup \{0\}$. Let us define a function m from $N \times F$ to $\bar{C}$ by $m(n, f) = nf$. The function m is continuous and is clearly onto, since any element in $\bar{C}$ is on a ray Nf, where $f \in F$. Also, it is easily seen that m is one-to-one except that $m(0, f) = m(0, g)$, for all $f, g \in F$. Now, let us consider the following diagram.

Since the P rays are a continuous collection, it follows that m^* is an iseomorphism. Let $\Phi(P)$ be the image of P in G'. Then there is a sector of one dimensional groups in S', namely $\Phi(P) \cdot \Psi(F)$. Furthermore, this sector is a maximal sector of one dimensional groups, otherwise $P \cdot F$ would not be a maximal sector of one dimensional groups in S. As above, the closure of the sector $\Phi(P) \cdot \Psi(F)$ is iseomorphic to $[\Phi(N) \times \Psi(F)]/[\{0\} \times \Psi(F)]$ under m^*. Now. let us define a function from $(N \times F)$ to $[\Phi(N) \times \Psi(F)]$ by choosing this function to be $\Phi \times \Psi$. This induces $(\Phi \times \Psi)^*$ from the factor sets which

$$
\begin{array}{ccc}
\dfrac{N \times F}{\{0\} \times F} & \xrightarrow{\quad m^* \quad} & C \\
\Big\uparrow \delta & & \Big\uparrow \text{identity} \\
N \times F & \xrightarrow{\quad m \quad} & C
\end{array}
$$

is clearly an iseomorphism. Hence, the sector $C = P \cdot F$ is iseomorphic to the sector $\Phi(P)\Psi(F)$ by the restriction of the map $\Phi \cdot \Psi$, defined by $(\Phi \cdot \Psi)(pf) = \Phi(p)\Psi(f)$, which is another way to describe $(\Phi \times \Psi)^*$. Let us call the sector $\Phi(P)\Psi(F)$, C'. Let us define λ from $\bar{G} \cup \bar{C}$ into $\bar{G}' \cup \bar{C}'$ by $\lambda(x) = \Phi(x)$, if $x \in G$, and $\lambda(x) = (\Phi \cdot \Psi)(x)$, if $x \in C$. Here, by $(\Phi \cdot \Psi)(x)$ we mean $(\Phi \cdot \Psi)(pf)$ where p, f are the unique elements in P, F respectively such that $x = pf$. It follows easily that λ is iseomorphism, and since there are only finitely many sectors of one dimensional groups, we can extend Φ to all such sectors in this manner.

Now, if S and S' are positive commutative Clifford semigroups on E^2 subject to the further hypotheses of this theorem, the iseomorphism Φ from $\bar{H}(1)$ to $\bar{H}(1')$ can be extended to an iseomorphism α from S onto S' by defining $\alpha(x) = \Phi_k(x)$, where $x \in H(k)$, if x is not in a sector of one dimensional groups, and $\alpha(x) = \lambda(x)$ if x is in a sector of one dimensional groups. Let C_o and D_o be two dimensional identity components of maximal groups in S distinct from G and which share a bounding ray M. Then, if Φ_j takes $\bar{G} \cup \bar{C}_o$ to $\bar{G}' \cup \bar{C}_o'$ and Φ_k takes $\bar{G} \cup D_o$ to $\bar{G}' \cup D_o'$, because $\Phi_{j|M} = \Phi_{k|M}$, it follows as in Case (i) that $\bar{C}_o'$ and D_o' share the ray M'. In virtue of the cases which have been considered α is one-to-one and continuous from E^2 onto E^2 and is hence a homeomorphism. Finally, let us show that α preserves multiplication. Let $x, y \in S$ such that $x \in H(j)$ and $y \in H(k)$. Then, $x = ja$ and $y = kb$ for some $a, b \in H(1)$. So, $xy = jkab$. But, jk is the idempotent element of the group $H(jk)$, and $(jkab) \in H(jk)$. Hence, $\alpha(xy) = \alpha(jkab) = j'k'\Phi(ab) = [j'\Phi(a)] \cdot [k'\Phi(b)] = [\Phi_j(ja)] \cdot [\Phi_k(kb)] = \Phi_j(x)\Phi_k(y) = \alpha(x)\alpha(y)$. Similarly, it is not difficult to show that α preserves multiplication on sectors of one dimensional groups. Thus, the theorem is established.

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