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SIMULTANEOUS COLOURING OF 4-VALENT MAPS

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In the present paper we consider maps on the sphere (in the sequel simply maps). We suppose that every such a map M has the following properties: (i) its graph is planar and finite, (ii) every face of M is incident with at least three edges, (iii) every vertex of M has degree four, (iv) every edge of M is adjacent to exactly two faces. Clearly elements of every 4-valent polyhedron form such a map.

We consider simultaneous colouring of vertices, edges and faces (in the sequel elements) of such maps. We shall consider various requirements concerning such a colouring. To express these requirements more briefly we define a matrix

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}$$

consisting of zeros and ones: $a_{ij} = 0$ (or $a_{ij} = 1$, respectively, for $i \neq j$) if incident elements of M of dimensions i and j may (or may not) have the same colour. For $i = j$ we put $a_{ij} = 1$ if adjacent elements of M of dimension i may not have the same colour; in other cases $a_{ij} = 0$. Clearly $a_{ij} = a_{ji}$. The above matrix can be written as follows:

$$(a_{00} \quad a_{01} \quad a_{02} \quad a_{11} \quad a_{12} \quad a_{22})$$

(we shall call such a sequence a colouring scheme).

By $\chi(a_{00}\ a_{01}\ a_{02}\ a_{11}\ a_{12}\ a_{22})$, or briefly by $\chi(a_{ij})$, we mean the minimal number of colours necessary for colouring all the elements of any 4-valent map M on the sphere if the colouring scheme is $(a_{00}\ a_{01}\ a_{02}\ a_{11}\ a_{12}\ a_{22})$.

An impulse for the present study was given by the paper of Jucovič [4]. Neuberger investigated an analogous problem for 3-valent maps on the sphere (see Izbicki [3]). Ringel [6] considers a special case of simultaneous colouring on the sphere. Table 1 shows for every colouring scheme (a_{ij}) (whose number is given in column N) an upper bound $\chi(a_{ij})$, where some estimates are strict.

Proofs of statements about colouring schemes for the numbers 0, 2, 8, 10, 16, 18, 24, 26 are trivial. Proofs of statements about colouring schemes for

Table 1

N	(a_{ij})	$\chi(a_{ij})$	N	(a_{ij})	$\chi(a_{i\cdot})$
0	(000 00 0)	1	32	(100 00 0)	4
1	(000 00 1)	2	33	(100 00 1)	4
2	(000 01 0)	2	34	(100 01 0)	4
3	(000 01 1)	3	35	(100 01 1)	4
4	(000 10 0)	5	36	(100 10 0)	5
5	(000 10 1)	5	37	(100 10 1)	5
6	(000 11 0)	$\leqslant 6$	38	(100 11 0)	$\leqslant 6$
7	(000 11 1)	$\leqslant 7$	39	(100 11 1)	$\leqslant 7$
8	(001 00 0)	2	40	(101 00 0)	$\leqslant 5$
9	(001 00 1)	3	41	(101 00 1)	$\leqslant 6$
10	(001 01 0)	2	42	(101 01 0)	$\leqslant 5$
11	(001 01 1)	3	43	(101 01 1)	$\leqslant 6$
12	(001 10 0)	5	44	(101 10 0)	5
13	(001 10 1)	5	45	(101 10 1)	$\leqslant 6$
14	(001 11 0)	$\leqslant 6$	46	(101 11 0)	$\leqslant 6$
15	(001 11 1)	$\leqslant 7$	47	(101 11 1)	$\leqslant 7$
16	(010 00 0)	2	48	(110 00 0)	$\leqslant 5$
17	(010 00 1)	2	49	(110 00 1)	$\leqslant 5$
18	(010 01 0)	2	50	(110 01 0)	$\leqslant 5$
19	(010 01 1)	3	51	(110 01 1)	$\leqslant 5$
20	(010 10 0)	5	52	(110 10 0)	$\leqslant 8$
21	(010 10 1)	5	53	(110 10 1)	$\leqslant 8$
22	(010 11 0)	$\leqslant 6$	54	(110 11 0)	$\leqslant 9$
23	(010 11 1)	$\leqslant 7$	55	(110 11 1)	$\leqslant 10$
24	(011 00 0)	2	56	(111 00 0)	$\leqslant 5$
25	(011 00 1)	3	57	(111 00 1)	$\leqslant 6$
26	(011 01 0)	3	58	(111 01 0)	$\leqslant 6$
27	(011 01 1)	4	59	(111 01 1)	$\leqslant 7$
28	(011 10 0)	$\leqslant 6$	60	(111 10 0)	$\leqslant 9$
29	(011 10 1)	$\leqslant 7$	61	(111 10 1)	$\leqslant 10$
30	(011 11 0)	$\leqslant 6$	62	(111 11 0)	$\leqslant 9$
31	(011 11 1)	$\leqslant 7$	63	(111 11 1)	$\leqslant 10$

52, 53, 55, 61, 62 and 63 follow from the proof of Theorem 3 by Jucovič [4 Proofs of the rest of the statements are contained in our Theorems 1, 2, 3, or are their consequences.

Theorem 1. $\chi(000\ 10\ 0) = 5$.

Proof. See Vizing [7] or Fiamčík and Jucovič [2].

It is not possible to improve the bound five. This is clear from the following statement: If a 4-valent map M has an odd number of vertices, then for colouring its edges according to (000 10 0) it is necessary to have exactly five colours. For, suppose that we can, colour according to (000 10 0) the edges of the map M, which has an odd number of vertices, by four colours $\alpha, \beta, \gamma, \delta$. We delete from M all edges coloured, e. g., by the colour α. The map M_1 formed in this way from M is 3-valent and has an odd number of vertices. It is a contradiction to the well-known property that the number of vertices of odd degree in every map is even.

Statements 4, 6, 12, 14, 28 of Tab. 1 are consequences of Theorem 1, which is easily shown. The colouring scheme (011 10 0), e. g., demands that incident vertex and edge, vertex and face and also adjacent two edges have different colours. According to Theorem 1, the colours $\alpha, \beta, \gamma, \delta, \varepsilon$ are sufficient for colouring edges. We colour the vertices by the colour ω. We colour the faces by the colours $\alpha, \beta, \gamma, \delta, \varepsilon$. It follows that $\chi(011\ 10\ 0) \leqslant 6$.

Theorem 2. (Brooks [1]). *If all the vertices of a connected graph G are of valence $< k$, then either G is k-colourable, or G is the complete graph with $k + 1$ vertices.*

From Theorem 2 it follows that to colour the vertices with respect to the colouring scheme (100 00 0) it is sufficient to have four colours for every 4-valent maps (the map formed by the complete graph with five vertices cannot be realized on the sphere). It is not possible to improve the bound four, because for every even integer $N \geqslant 8$ not divisible by 6 we can construct a 4-valent map M' with exactly N vertices such that for colouring its vertices according to (100 00 0) it is necessary to have exactly four colours.

Let us prove that the map M' drawn in Fig. 1 has such properties. Suppose that it is 3-colorable. We shall consider in M' the circuit

$$\mathscr{K} = x_1, h_1, x_2, \ldots, x_{i-1}, h_{i-1}, x_i, h_i, x_{i+1}, \ldots, x_1 .$$

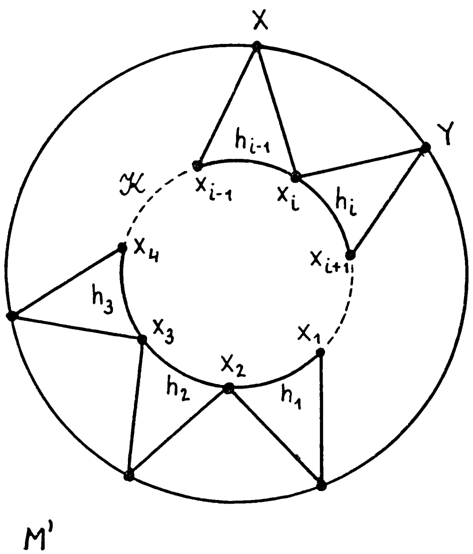

Fig. 1

The vertices of $\mathscr{K}$ can be coloured by two colours α, β or by three ones α, β, γ. If vertices of $\mathscr{K}$ are alternatively coloured by two colours α, β, *then the* vertices $x_{i-1}, x_{i+1} \in \mathscr{K}$ must have the same colour, e. g. α. *Then* x_i *must be* coloured by β. As the vertex X is adjacent *with* x_{i-1}, x_i, *the vertex* X must be coloured by γ. As vertex Y is adjacent *with vertices* x_i, x_{i+1}, X, *the vertex* Y must not be coloured by any colour from α, β, γ.

Let vertices of $\mathscr{K}$ are coloured by three colours. As the number of vertices of M' is not divisible by 6, then the number of vertices of $\mathscr{K}$ is not divisible by 3. Then at every colouring of vertices of $\mathscr{K}$ by three colours there exist in $\mathscr{K}$ two vertices x_{i-1}, x_{i+1}, which have the same colour. Next we process analogously as above. Thus we showed that for colouring vertices of M' according to (100 00 0) we need exactly four colours.

Proofs of statements 32, 34, 40, 42, 48, 50, 56, 58 follow from Theorem 2; we show it e. g. for the colouring scheme (110 01 0). According to Theorem 2 the vertices of M can be coloured by colours $\alpha, \beta, \gamma, \delta$. Since incident edge and vertex, as well as face and edge need different colours, we colour edges by a colour ε. The faces of M we colour by colours $\alpha, \beta, \gamma, \delta$. Therefore $\chi(110\ 01\ 0) \leqslant 5$.

Theorem 3. *A finite planar graph G has a face coloration in two colours if, and only if, all its valences are even, that is, the components of G are Euler graphs*

Proof. See Ore [5], p. 77.

From Theorem 3 it follows that two colours are sufficient for colouring any 4-valent map M on the sphere according to (000 00 1).

This gives proofs of statements 1, 3, 9, 11, 17, 19, 25, 27 of Tab. 1. We shall show it for number 27. According to the colouring scheme (011 01 1) a vertex and an edge incident with it need different colours. We colour vertices by a colour α, edges by a colour β. No face and incident vertex (edge) can be coloured in the same way. Two adjacent faces need different colouring, too. We colour faces of M by colours γ, δ. It follows that $\chi(011\ 01\ 1) = 4$.

We can colour edges of M by five colours according to the colouring scheme (000 10 0) (according to Theorem 1). As every vertex of M is incident with exactly four edges, we can use the fifth colour from $\alpha, \beta, \gamma, \delta, \varepsilon$ to colour the vertex that is incident with these edges. We can colour edges and vertices of M by this method with colouring schemes 20, 22, 30 from Tab. 1.

In a map M there can exist such a face f, whose vertices (edges) are coloured by all colours which are used for colouring vertices (edges) of M. Therefore if every vertex (edge) which is incident with a face f has a different colour, then we need another colour for the face f. The colouring scheme (100 11 1) demands e. g. that incident face and edge have different colours and the same colours for adjacent vertices, edges and faces. To colour the edges, the colours

$\alpha, \beta, \gamma, \delta, \varepsilon$ (according to Theorem 1) are sufficient. Since incident vertex and edge can have the same colour, we can colour these vertices by four colours from $\alpha, \beta, \gamma, \delta, \varepsilon$ (according to Theorem 2). We colour the faces by the colours ω, τ. It follows that $\chi(100\,11\,1) \leqslant 7$. By analogous methods we can prove the rest of the statements in Tab. 1.

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