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ON THE FRATTINI IDEAL IN A CERTAIN CLASS OF SEMIGROUPS

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Let S be a semigroup and consider the law $xyzyx = yxzxy$ for all $x, y, z \in S$. The commutative semigroups and all subsemigroups of class 2 nilpotent groups (see Neumann-Taylor [3]) satisfy this law. Accordingly, this law will be called the C_2 law and any subsemigroup of a group which satisfies it will be referred to as a class 2 semigroup.

The Frattini ideal, $\Phi(S)$, of a semigroup S is the intersection of all the maximal ideals of S. According to S. Schwarz [4], $\Phi(S)$ is always non-empty, provided that S has maximal ideals. A. D. Kacman [2] and, more recently, P. A. Grillet [1] and S. Schwarz [4] discuss several aspects of this ideal in their respective papers and the reader is referred to their articles for relevant definitions. Kacman concentrates his efforts on invariant subsemigroups of groups, which, of course, includes the commutative case.

The purpose of this note is to expand on some of Kacman's existence theorems by showing that the Frattini ideal exists: (1) if S is a finitely generated, pure, class 2 semigroup and (2) if S is a centerless, class 2 semigroup with a minimal (though not necessarily finite) set of generators. It may be recalled that a pure subsemigroup of a group is a semigroup without pairs of mutually inverse elements.

1. Theorem. *Let S be a finitely generated, pure, class 2 semigroup. Then, S possesses a maximal ideal.*

Proof. Let $S = \text{sem}\,\{s_1, \ldots, s_n\}$ be the semigroup generated by the elements $s_1, \ldots, s_n$ of S. Consider the ideal generated by s_1^3, $i(s_1^3)$. It will be shown that $i(s_1^3)$ is a proper ideal of S. If $i(s_1^3)$ is not a proper ideal of S, then $s_1 \in i(s_1^3)$. s_1 is of the form ss_1^3, s_1^3s or ss_1^3t for some $s, t \in S$. Using the last representation of s_1 as an element of $i(s_1^3)$, the contradiction that 1, the identity element of the group, belongs to S will be drawn. The other representations of s_1 as an element of $i(s_1^3)$ require similar arguments. Thus, suppose $s_1 = ss_1^3t$ for some $s, t \in S$. Then by the C_2 law, $s_1^3 = ss_1^3t \,.\, ss_1^3t \,.\, ss_1^3t = ss_1^3 \,.\, ss_1^3\, t^3ss_1^3$ so that $1 = ss_1^3ss_1^3t^3s$ and 1 belongs to S. Since S is pure, this is not possible.

The above argument proves the existence of a proper ideal J which does not meet s_1. Let $K_1 \supseteq J$, where K_1 is an ideal maximal with respect to the property of not containing s_1, and consider $i(K_1, s_1) = J_1$, the ideal generated by $K_1 \cup \{s_1\}$. If $J_1 = S$, then K_1 is the maximal ideal sought. If not, choose $K_2 \supseteq J_1$, where K_2 is an ideal maximal with respect to the property of not containing t_2, the first of $s_2, \ldots, s_n$ not contained in J_1. Since $S = \text{sem }\{s_1, \ldots, s_n\}$, by continuing this process a $K_j \supseteq J_{j-1} \supseteq \ldots \supseteq J$ must be reached where $i(K_j, t_j) = S$ and K_j is the maximal ideal sought.

Remark. The following statement, Theorem 2, comes as a consequence of Theorem 1 and the previously mentioned theorem of Schwarz. However, an alternate proof for Theorem 2 is given which also illustrates what $\Phi(S)$ is equal to.

2. Theorem. *If S is a finitely generated, pure, class 2 semigroup, then $\Phi(S)$ is non-empty.*

Proof. Since S is finitely generated, there exists a set $\beta = \{b_j \mid j = 1, \ldots, n\}$ which generates S. As in the proof of Theorem 1, $i(b_j^3)$ avoids b_j. Thus, for each j, there exists an ideal M_j, maximal in that it avoids b_j. According to Theorem 1, S possesses at least one maximal ideal. Any maximal ideal avoids some b_j and consequently is equal to M_j. Let $J = \{j' \mid j'$ is some j for which M_j is maximal$\}$ and $\beta' = \{b_{j'} \mid j' \in J\}$. $\Phi(S) = \bigcap_{j' \in J} M_{j'}$. It will be shown that $\Phi(S)$ is not empty and, in fact, exhibit what it is equal to. $i(\prod_{j' \in J} b_{j'}^3)$ is, as in the proof of Theorem 1, an ideal which avoids β'. This implies the existence of an ideal M_Φ which is maximal with respect to the property of avoiding β'. Clearly, $M_\Phi \subseteq \Phi(S)$. On the other hand, for any $x \in \Phi(S)$, $i(x) \subseteq M_{j'}$, for each $j' \in J$ and, consequently, $b_{j'} \notin i(x)$ for each $j' \in J$. Thus, $i(x) \subseteq M_\Phi$ and $x \in M_\Phi$. In view of the above, $\Phi(S) = M_\Phi$.

3. Theorem. *Let S be a centerless, class 2 semigroup with a minimal set of generators β. Then $\Phi(S) = S - \beta$.*

Proof. Let $\beta = \{b_\alpha \mid \alpha \in A\}$. Each b_α is non-decomposable in S, that is, b_α cannot be presented as a product of two elements in S. For, suppose $b \in \beta$ and $b = st$, with $s, t \in S$. Since b is a member of a minimal generating set, s or t must contain b as a factor. If $s = bs_1$ or $t = t_1 b$, with $t_1, s_1 \in S \cup \{1\}$, then S possesses an identity element, contrary to hypothesis. On the other hand, suppose $b = st = s_1 b t_1$, with $s_1, t_1 \in S$. Then, $b^{-1}s_1^{-1}bs_1 = t_1 s_1 \in S$. Since S is a class 2 semigroup, it generates a class 2 group (see Neumann—Taylor [3]) so that the commutator $b^{-1}s_1^{-1}bs_1$ is central in the group generated by S. Thus, S possesses a central element, also contrary to hypothesis.

Since b_α is non-decomposable, $S - b_\alpha$ is clearly an ideal of S, and all maximal ideals are of such a form. It is now apparent that $\Phi(S)$ exists and is equal to $S - \beta$.

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