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Matematický časopis, Vol. 22 (1972), No. 1, 59--70

Persistent URL: http://dml.cz/dmlcz/126789

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SOME CHARACTERIZATIONS OF THE DARBOUX CONTINUITY OF REAL FUNCTIONS

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1. Introduction

In recent years a number of articles appeared which deal with the limits of sequences of Darboux functions (we consider real-valued Darboux functions defined on the real line). It is known that the limit function of a sequence of Darboux functions may fail to be Darboux though the sequence converges uniformly (see the expository paper [1] of Bruckner and Ceder). The following problem has been stated by S. Marcus (see [1]): What is the „natural" type of convergence „$\Rightarrow$" for Darboux functions, i. e. what type of convergence „$\Rightarrow$" has the property that if $\{f_n\}_{n=1}^{\infty}$ is a sequence of Darboux functions converging pointwise to f then f is Darboux if and only if $f_n \Rightarrow f$ (i. e. when f_n converges to f in the sense of „$\Rightarrow$"). It is very difficult to describe such a type of convergence in general but in the present paper a „characteristic" type of convergence for uniformly converging sequences of Darboux functions is given (see Theorem 2 below). It is shown that the real-valued Darboux functions defined on the real line can be characterized as the continuous functions from one topological space to another topological space (Theorem 1 below). There are also given some types of convergence which preserve the Darboux continuity (see Theorems 3 and 4 below).

In the sequel, the set of real numbers is denoted as R_0 while the set $R_0 \cup \cup \{-\infty\} \cup \{+\infty\}$ of extended real numbers as R. $\mathscr{D}$ stands for the class of Darboux functions. The fact that f is a function with a domain A and a range B is written as $f: A \to B$.

2. Preliminary Constructions

Let S be the cartesian product $S = R_0 \times \{-, +\}$ of the set R_0 of real numbers ordered by the usual order-relation, and the set $\{-, +\}$ whose only elements are the symbols $-$ and $+$ ordered by $- < +$. If (a, α) is an element of S, then $a \in R_0$ is called the real part of (a, α), and $\alpha \in \{-, +\}$

the characteristic of (a, α). Assume S to be ordered by the lexocigraphic relation defined as follows: If (a, α) and (b, β) are two elements of S then $(a, \alpha) < (b, \beta)$ if and only if $a < b$, or $a = b$ and $\alpha < \beta$. Let $\mathscr{T}$ be the order topology for S generated by this ordering.

It is easy to verify that $(S, \mathscr{T})$ is a first countable topological space (i. e. the neighbourhood system of every its point has a countable base). The space $(S, \mathscr{T})$ is also separable and does not satisfy the second axiom of countability. Hence $(S, \mathscr{T})$ fails to be a metric space (see Kelley [3]).

The following lemmas give more information on the structure of the topological space $(S, \mathscr{T})$.

Lemma 1. *Each non-empty bounded subset M of S has the least upper bound.*

Proof: Assign to each element x of M its real part x'. The set M' of all this elements x' has the (real) least upper bound y'. Now let $y = (y', -)$ if $(y', +) \notin M$, and $y = (y', +)$ if $(y', +) \in M$. It is easy to verify that y is the least upper bound of M, q. e. d.

Lemma 2. *Every closed bounded subinterval I of S is a compact.*

Proof: Let I be some closed bounded subinterval of S with the end-points a, b, $a < b$, and let $\mathscr{G} \subset \mathscr{T}$ be an open cover of I. We may assume without loss of generality that the characteristic of a is $+$, and the characteristic of b is $-$. We wish to show that there is a finite subfamily of $\mathscr{G}$ which covers the interval I.

Denote by A the set of all elements $x \in I$ such that the closed interval $\langle a, x\rangle = \{y \in I;\ a \leqslant y \leqslant x\}$ has a finite subcover. Clearly $a \in A \neq \emptyset$. Let s be the least upper bound of A, and let s' be the real part of s. Then the interval $\langle a, (s', -)\rangle$ has a finite subcover. To see it we may assume that $a < (s', -)$. The point $(s', -)$ is in some open set $G \in \mathscr{G}$, hence G contains some open interval $\langle(s' - \varepsilon, +), (s', -)\rangle$, where $\varepsilon > 0$ is sufficiently small. Since $(s' - \varepsilon, +) < s$, the interval $\langle a, (s' - \varepsilon, +)\rangle$ has a finite subcover and hence $\langle a, (s', -)\rangle = \langle a, (s' - \varepsilon, +)\rangle \cup \langle(s' - \varepsilon, +), (s', -)\rangle$ has also a finite subcover. Now if $s < b$, then the point $(s', +)$ is in some $G' \in \mathscr{G}$, hence G' contains an open interval $\langle(s', +), (s' + \varepsilon', -)\rangle$ with a sufficiently small $\varepsilon' > 0$. Since $\langle a, (s', -)\rangle$ has a finite subcover the interval $\langle a, (s' + \varepsilon', -)\rangle = \langle a, (s', -)\rangle \cup \langle(s', +), (s' + \varepsilon', -)\rangle$ has also a finite subcover contrary to the fact that $s < (s' + \varepsilon', -)$. Lemma 2 is proved.

Lemma 3. *Each non-empty closed subset P of S is a second category set in itself.*

Proof: Let $X = \bigcup_{i=1}^{\infty} P_i$, where P_i are nowhere dense in P. We wish to show that $P - X \neq \emptyset$. Since P_1 is nowhere dense in P there is an open interval I

such that $I \cap P \neq \emptyset$ and $I \cap \bar{P}_1 = \emptyset$($\bar{A}$ denotes the closure of A). It is easy to verify that I contains a closed bounded interval J_1 such that $(\operatorname{int} J_1) \cap \cap P \neq \emptyset$. Assume by induction that the closed intervals J_k, $1 \leqslant k < n$, have been constructed such that

$$J_1 \supset J_2 \supset \ldots \supset J_{n-1}, \ (\operatorname{int} J_{n-1}) \cap P \neq \emptyset, \quad \text{and} \quad J_k \cap \bar{P}_k = \emptyset,$$

for every k, $1 \leqslant k < n$. Since P_n is nowhere dense in P, the set int J_{n-1} contains some closed interval J_n such that $(\operatorname{int} J_n) \cap P \neq \emptyset$ and $J_n \cap \bar{P}_n = \emptyset$. Now, by Lemma 2, the interval J_1 is a compact, and $\{J_n \cap P\}_{n=1}^{\infty}$ is a family of closed subsets of J_1 which have the finite intersection property, hence (see Kelley [3], p. 136) the set $\bigcap_{n=1}^{\infty} (J_n \cap P) = (\bigcap_{n=1}^{\infty} J_n) \cap P$ is non-empty and $(\bigcap_{n=1}^{\infty} J_n) \cap P \subset P - X$, q. e. d.

Next consider another topological space. Let $\mathscr{F}$ be the family of closed subintervals of $R = R_0 \cup \{-\infty\} \cup \{+\infty\}$, and let $\mathscr{T}_1$ be a topology for $\mathscr{F}$ with the following base $\mathscr{B}$: $G \in \mathscr{B}$ if and only if there is an open set G_1 in R such that $G = \{A \in \mathscr{F};\ A \subset G_1\}$. Clearly $(\mathscr{F}, \mathscr{T}_1)$ is a compact.

Let $f : R_0 \to R_0$ be a function. The left range $R_f(x, -)$ of f in x, and the right range $R_f(x, +)$ of f in x are the sets

$$R_f(x, -) = \bigcap_{n=1}^{\infty} f\left(\left(x - \frac{1}{n}, x \right\rangle\right)$$

and

$$R_f(x, +) = \bigcap_{n=1}^{\infty} f\left(\left\langle x, x + \frac{1}{n} \right)\right),$$

respectively. Clearly $f(x) \in R_f(x, -) \cap R_f(x, +)$.

Now to each function $f : R_0 \to R_0$ assign three functions

$$f_* : S \to R, \qquad f^* : S \to R, \quad \text{and} \quad \tilde{f} : S \to \mathscr{F}$$

defined as follows: If $I = \langle a, b \rangle$ is the closure of the connected component of a set $R_f(x, -)$ (resp. $R_f(x, +)$), which contains the point $f(x)$, then

$$f_*(x, -) = a, \qquad f^*(x, -) = b, \quad \text{and} \quad \tilde{f}(x, -) = I$$

$$(\text{resp. } f_*(x, +) = a, \qquad f^*(x, +) = b, \quad \text{and} \quad \tilde{f}(x, +) = I).$$

The functions $\tilde{f}$ play an essential role in the next sections.

3. A Characterization Theorem for Darboux Functions

The following two lemmas show that if $f : R_0 \to R_0$ is a Darboux function, then the functions f_* and f^* have characteristic properties.

Lemma 4. *For each Darboux function $f: R_0 \to R_0$, f_* is a lower semi-continuous function, and f^* is an upper semi-continuous function.*

Proof: We prove the Lemma for f_* (for f^* the proof is similar). Let $z \in [f_* > \lambda]$. Since the construction is symmetric we may assume the characteristic of z to be $-$, i. e. $z = (z', -)$. Hence $f_*(z) > \lambda$. Choose a λ' such that $f_*(z) > \lambda' > \lambda$. Since f is a Darboux function, the set $R_f(z) = R_f(z', -)$ is connected (see Bruckner and Ceder [1]) and hence $f_*(z) = f_*(z', -) = \inf R_f(z', -)$; thus $\lambda' < \xi$ for every $\xi \in R_f(z', -) = \bigcap_{n=1}^{\infty} f((z' - 1/n, z'\rangle)$, and since every set $f((z' - 1/n, z'\rangle)$ is connected, there is some n_0 such that $\lambda' < \zeta$ for every $\zeta \in f\left(\left(z' - \frac{1}{n_0}, z'\right\rangle\right)$. Now for each $y \in \left(z' - \frac{1}{n_0}, z'\right)$, $R_f(y, +) \subset f\left(\left(z' - \frac{1}{n_0}, z'\right\rangle\right)$ and $R_f(y, -) \subset f\left(\left(z' - \frac{1}{n_0}, z'\right\rangle\right)$ hence, for each such y we have

$$\inf R_f(y, +) = f_*(y, +) \geqslant \lambda' > \lambda$$

and

$$\inf R_f(y, -) = f_*(y, -) \geqslant \lambda' > \lambda .$$

Thus the set $[f_* > \lambda]$ contains an open neighbourhood $((z' - 1/n_0, +), (z', -)\rangle$ of $z = (z', -)$ which proves the set $[f_* > \lambda]$ to be open, q. e. d.

Lemma 5. *For each function $f: R_0 \to R_0$, if f_* is lower semi-continuous, and f^* upper semi-continuous, then f is a Darboux function.*

Proof: Let f_* be lower semi-continuous and f^* upper semi-continuous. Assume that contrary to what we wish to show there are numbers $x_1 < x_2$ and c such that $f(x_1) < c < f(x_2)$ and $f(\xi) \neq c$ for every $\xi \in \langle x_1, x_2\rangle$ (for $f(x_1) > f(x_2)$ the proof is similar). Let $A = [f > c] \cap (x_1, x_2)$ and $B = [f < c] \cap (x_1, x_2)$. Both the sets A and B are bilaterally dense in itself. To see it assume that there is a point $x_0 \in A$, and some $\varepsilon > 0$ such that $A \cap (x_0, x_0 + \varepsilon) = \emptyset$. In this case we have $f_*(x_0, +) = f(x_0) > c$ but $f_*(z) < c$ for each $z \in ((x_0, +), (x_0 + \varepsilon, -))$, hence $(x_0, +)$ cannot be an interior point of $[f_* > c]$ and consequently f_* fails to be lower semicontinuous. Thus we have proved that every point of A is a cluster point of every its right-hand neighbourhood. In other cases the proof is similar.

Thus the connected components of the sets A, B are closed intervals. Let $\mathscr{M}$ be the set of components of A and B which contain more than one point, i. e. of components of the form $K = \langle x, y\rangle$, $x < y$. To every such component K

assign the set $K' = \{(x, y)\} \times \{-, +\} \cup \{x\} \times \{+\} \cup \{y\} \times \{-\}$. Clearly, K' is an open set (in $(S, \mathscr{T})$). Now put

$$P = \{(x_1, x_2)\} \times \{-, +\} - \bigcap_{K \in \mathscr{M}} K'.$$

The interval (x_1, x_2) cannot be written as the union of a (at most countable) family of pairwise disjoint closed nontrivial intervals (Sierpiński [4], p. 220—221), hence there are components of A or B which contain exactly one point. From this it follows that P is non-empty. The set P is also closed. Now let $P = P_1 \cup P_2$, where P_1 is the set of $z \in P$ with real part in A, and P_2 the set of $z \in P$ with real part in B. Both the sets P_1 and P_2 are dense in P, i. e.

$$\overline{P}_1 = \overline{P}_2 = P. \tag{1}$$

Indeed, let $z \in P$ and assume $z = (z', -)$, where $z' \in A$ (in other cases the proof is similar). Since $z' \in A$, we have $z \in \overline{P}_1$. On the other hand $z \in P$, hence the point z' cannot be the right-hand end-point of any non-trivial component of the set A; thus in every left-hand neighbourhood of z' there is a point of B. But in this case every left-hand neighbourhood of $z = (z', -)$ contains some point of P_2, hence $z \in \overline{P}_2$.

Since P is closed f_* is lower semi-continuous, and f^* is upper semi-continuous, each of the sets

$$\left[f_* \leqslant c - \frac{1}{n}\right] \cap P, \qquad \left[f^* \geqslant c + \frac{1}{n}\right] \cap P, \qquad n = 1, 2, \ldots,$$

is closed. There is also

$$\left[f_* \leqslant c - \frac{1}{n}\right] \cap P \subset P_2 \tag{2}$$

and

$$\left[f^* \geqslant c + \frac{1}{n}\right] \cap P \subset P_1; \tag{3}$$

indeed, if $f_*(z) \leqslant c - \dfrac{1}{n}$ and (say) $z = (z', +)$, then $f(z') < c$ hence $z \in P_2$ (similarly for f^*). Now from (1) it follows that each of the sets (2) and (3) is nowhere dense in P. But

$$P = \left(\bigcup_{n=1}^{\infty} \left[f_* \leqslant c - \frac{1}{u}\right] \cap P\right) \cup \left(\bigcup_{n=1}^{\infty} \left[f^* \geqslant c + \frac{1}{n}\right] \cap P\right),$$

hence P is a set of the first category in itself contrary to the fact that P is closed and non-empty (see Lemma 3). Thus Lemma 5 is proved.

The next theorem is a consequence of Lemmas 4 and 5 and gives a characterization of Darboux functions using the notion of continuity.

Theorem 1. *Let $f: R_0 \to R_0$. Then f is Darboux if and only if $\tilde{f}$ is continuous.*

Proof: It is easy to see that $\tilde{f}$ is continuous if and only if f_* is lower semi-continuous and f^* upper semi-continuous. From this and from Lemmas 4 and 5 the theorem follows.

4. A Characteristic Type of Convergence for Uniformly Converging Sequences of Darboux Functions

The following Theorem 2 gives a characteristic type of convergence for uniformly converging sequences of Darboux functions. (For facts concerning the uniform closure of $\mathscr{D}$ see Bruckner, Ceder and Weiss [2]). In this section and in Section 5 we use this convention: If $x, y \in R$, and $\varepsilon \in R_0$, $\varepsilon > 0$, then $|x - y| < \varepsilon$ if and only if $x, y \in R_0$ and $|x - y| < \varepsilon$ in the usual sense, or $x = y = +\infty$, or $x = y = -\infty$. Cauchy sequences and uniformly converging sequences of functions with R as domain must be interpreted similarly.

To prove the theorem the following three lemmas are necessary.

Lemma 6. *Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence of Darboux functions $f_n : R_0 \to R_0$. Then both $\{f_n^*\}_{n=1}^{\infty}$, and $\{f_{n*}\}_{n=1}^{\infty}$ are Cauchy sequences.*

Proof: Because of symmetry of the construction it suffices to prove that there is some n_0 such that $m > n_0$ implies $|f_{n_0}^*(z) - f_m^*(z)| < \varepsilon$ for arbitrary $z \in S$ with characteristic $+$ $(z = (z', +))$ (for $\{f_{n*}\}_{n=1}^{\infty}$, and for $z = (z', -)$ the argument is similar).

Each f_n is in $\mathscr{D}$, hence for every positive integers n, k, the set $f_n\left(\left\langle z', z' + \frac{1}{k}\right)\right)$ is an interval and since $f_n\left(\left\langle z', z' + \frac{1}{k}\right)\right) \supset f_n\left(\left\langle z', z' + \frac{1}{k+1}\right)\right)$ we have

$$f_n^*(z) = \sup R_{f_n}(z) = \sup \bigcap_{k=1}^{\infty} f_n\left(\left\langle z', z' + \frac{1}{k}\right)\right) = \tag{4}$$

$$= \lim_{k \to \infty} \left(\sup f_n\left(\left\langle z', z' + \frac{1}{k}\right)\right)\right),$$

for every n. Let $\varepsilon > 0$. There is some n_0 such that, for each $x \in R_0$, $|f_{n_0}(x) -$

$-f_m(x)| < \varepsilon$ whenever $m > n_0$. For such m, n_0, from (4) it follows that

$$f_m^*(z) - \varepsilon = \lim_{k\to\infty} \left(\left(\sup f_m \left(\left\langle z', z' + \frac{1}{k}\right)\right)\right) - \varepsilon\right) \leqslant$$

$$\leqslant \lim_{k\to\infty} \left((\sup f_{n_0}\left(\left\langle z', z' + \frac{1}{k}\right)\right)\right) = f_{n_0}^*(z) \leqslant$$

$$\leqslant \lim_{k\to\infty} \left(\left(\sup f_m \left(\left\langle z', z' + \frac{1}{k}\right)\right)\right) + \varepsilon\right) = f_m^*(z) + \varepsilon.$$

Thus $|f_{n_0}^*(z) - f_m^*(z)| < \varepsilon$, whenever $m > n_0$, q. e. d.

Lemma 7. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Darboux functions $f_n : R_0 \to R_0$ converging uniformly to a function f. Then $\lim_{n\to\infty} (f_n)_* \leqslant f_*$, and $\lim_{n\to\infty} f_n^* \geqslant f^*$.*

Proof: We prove that $\lim_{n\to\infty} f_n^*(z) \geqslant f^*(z)$, where $z = (z', +)$ (for $\lim_{n\to\infty} (f_n)_* \leqslant f_*$, and for $z = (z', -)$ the argument is similar). Let $\varepsilon > 0$. There is some n_0 such that $f_n + \varepsilon > f$, whenever $n > n_0$. For such n, using (4) we get

$$(5)\quad f_n^*(z) + \varepsilon = \lim_{k\to\infty} \left(\left(\sup f_n \left(\left\langle z', z' + \frac{1}{k}\right)\right)\right) + \varepsilon\right) \geqslant \lim_{k\to\infty} \left(\sup f\left(\left\langle z', z' + \frac{1}{k}\right)\right)\right).$$

It is easy to verify that

$$f^*(z) \leqslant \sup \bigcap_{k=1}^{\infty} f\left(z', z' + \frac{1}{k}\right)\right) \leqslant \lim_{k\to\infty} \left(\sup f\left(\left\langle z', z' + \frac{1}{k}\right)\right)\right).$$

From this and from (5) it follows that $f_n^*(z) + \varepsilon \geqslant f^*(z)$, which proves the Lemma.

In the proof of the next Lemma 8 we use this property of semi-continuous functions: The uniform limit of a sequence of lower semi-continuous functions defined on a first countable topological space X is lower semi-continuous (similarly with upper semi-continuity). Although this property must be known I have been unable to find a reference. The property follows simply from the fact that a function f on X is lower semicontinuous if and only if, for each $x \in X$, and each sequence $\{x_n\}_{n=1}^{\infty}$ of points in X which converges to x,

$$(6)\qquad \liminf_{n\to\infty} g(x_n) \geqslant g(x)$$

(see Kelley [3], pp. 72 and 101).

Lemma 8. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Darboux functions $f_n : R_0 \to R_0$ converging uniformly to a function f. Then f is Darboux if and only if both $\lim_{n\to\infty} f_n^* = f^*$, and $\lim_{n\to\infty} (f_n)_* = f_*$.*

Proof: Let $f \notin \mathscr{D}$. By Lemma 6, $\{f_n^*\}_{n=1}^{\infty}$ converges uniformly to a function $g : S \to R$; since every f_n^* is upper semi-continuous the function g is also upper semi-continuous. Similarly the sequence $\{(f_n)_*\}_{n=1}^{\infty}$ converges uniformly to a lower semi-continuous function h. But $f \notin \mathscr{D}$, hence by Lemma 5 either $f^* \neq g$, or $f_* \neq h$, which proves the first implication.

Now let $f \in \mathscr{D}$. Clearly, it suffices to prove that $\lim\limits_{n\to\infty} f_n^*(z) = f^*(z)$ for some $z = (z', +) \in S$ whose characteristic is $+$ (in other cases the proof is similar). Let $\varepsilon > 0$. Using (4) we get, for sufficiently large n,

$$f_n^*(z) + \varepsilon = \lim_{k\to\infty}\left(\left(\sup f_n\left(\left\langle z', z' + \frac{1}{k}\right)\right)\right) + \varepsilon\right) \geqslant$$

$$\geqslant \lim_{k\to\infty}\left(\sup f\left(\left\langle z', z' + \frac{1}{k}\right)\right)\right) = f^*(z) \geqslant$$

$$\geqslant \lim_{k\to\infty}\left(\left(\sup f_n\left(\left\langle z', z' + \frac{1}{k}\right)\right)\right) - \varepsilon\right) = f_n^*(z) - \varepsilon, \text{ q. e. d.}$$

Now we are able to prove the following

Theorem 2. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Darboux functions $f_n : R_0 \to R_0$ converging uniformly to a function f. Then f is Darboux if and only if $\lim\limits_{n\to\infty} \tilde{f}_n = \tilde{f}$ (in the topology $\mathscr{T}_1$).*

Proof: Let $f \in \mathscr{D}$. Let $z \in S$ and let G be an open neighbourhood (in $\mathscr{T}_1$) of $\tilde{f}(z)$. There is an open interval $J \subset R$ such that $\tilde{f}(z) = \langle f_*(z), f^*(z)\rangle \subset J$, and every closed subinterval of J is in G. By Lemma 8, $\lim\limits_{n\to\infty} f_n^* = f^*$, and $\lim\limits_{n\to\infty} (f_n)_* = f_*$; hence $\tilde{f}_n(z) = \langle (f_n)_*(z), f_n^*(z)\rangle \subset J$ and hence $\tilde{f}_n(z) \in G$, for sufficiently large n. Thus $\tilde{f}_n$ converges to $\tilde{f}$.

On the other hand let $f \notin \mathscr{D}$. By Lemma 8, there is either $\lim\limits_{n\to\infty} f_n^* \neq f^*$, or $\lim\limits_{n\to\infty} (f_n)_* \neq f_*$, hence by Lemma 7 either $\lim\limits_{n\to\infty} f_n^*(z) > f^*(z)$, or $\lim\limits_{n\to\infty} (f_n)_*(z) < f_*(z)$. So there is some open interval $J \subset R$ such that $\tilde{f}(z) = \langle f_*(z), f^*(z)\rangle \subset J$ and there is an n as large as we want such that $\langle (f_n)_*(z), f_n^*(z)\rangle \not\subset J$. Now the set G of closed subintervals of J is a neighbourhood of $\tilde{f}(z)$ such that there is some n arbitrary large with $\tilde{f}_n(z) \notin G$. Thus $\tilde{f}_n$ fails to converge to $\tilde{f}$, q. e. d.

5. Some Sufficient Conditions for a Limit of Darboux Functions to be a Darboux Function

Since $\mathcal{D}$ is not closed under the uniform limits (see Bruckner, Ceder and Weiss [2]) from Theorem 2 it follows that there is a sequence $\{f_n\}_{n=1}^{\infty}$ of Darboux functions such that $\lim_{n\to\infty} f_n = f$, but $\tilde{f}_n$ fails to converge to $\tilde{f}$. In the present section we shall consider the sequences $\{f_n\}_{n=1}^{\infty}$ of Darboux functions $f_n : R_0 \to R_0$ with the following property: There is a function f such that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f and $\tilde{f}_n$ to $\tilde{f}$. For such sequences some sufficient and necessary conditions for f to be in $\mathcal{D}$ are shown below. At first we note that in general $\tilde{f}_n \to \tilde{f}$ does not imply $f \in \mathcal{D}$ as it is shown in the following example.

Example. Define $f_n : R_0 \to R_0$ by

$$f_n(x) = \begin{cases} 1 + \frac{1}{n}\sin\left(\frac{1}{x}\right) & \text{if } 0 < x \leqslant \frac{1}{n\pi}, \\ \frac{\pi(1 - nx)}{\pi - 1} & \text{if } \frac{1}{n\pi} < x \leqslant \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} < x, \\ 1 & \text{if } x \leqslant 0, \end{cases}$$

and let $f(x) = 0$ for $x > 0$, and $f(x) = 1$ for $x \leqslant 0$. Clearly every f_n is in $\mathcal{D}$ and $\lim_{n\to\infty} f_n = f \notin \mathcal{D}$. On the other hand, $\tilde{f}_n(0, +) = \left\langle 1 - \frac{1}{n}, 1 + \frac{1}{n} \right\rangle$, and for $z \neq (0, +)$, $\tilde{f}_n(z) = f_n(z')$, where z' is the real part of z. Similarly for every z, $\tilde{f}(z) = f(z')$, where z' is the real part of z. Thus $\tilde{f}_n$ converge to $\tilde{f}$.

The next theorem gives a sufficient condition for the limit of a sequence of Darboux functions to be also Darboux.

For the sake of simplicity, if I is an interval in R, and $\varepsilon > 0$, let $O\varepsilon(I)$ denote the open ε-neighbourhood of I (in R).

Theorem 3. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Darboux functions $f_n : R_0 \to R_0$ converging pointwise to a function f, and let $\tilde{f}_n$ converge pointwise to $\tilde{f}$. If for every $\varepsilon > 0$, and every m, there is some $n > m$ such that*

(7) $$\tilde{f}(z) \subset \bigcup_{k=m+1}^{n} O_\varepsilon(\tilde{f}_k(z)),$$

for every $z \in S$, *then* f *is a Darboux function.*

Proof: Let $\delta > 0$, and $z_0 \in S$. Since $(f_n)_*$ converges to f_* there is some m_0 such that

(8) $$m' > m_0 \text{ implies } (f_{m'})_*(z_0) > f_*(z_0) - \frac{\delta}{3}.$$

Put in (7) $m = m_0$, and $\varepsilon = \frac{\delta}{3}$. Since $(f_i)_*$, $m < i \leqslant n$, are lower semi-continuous there is a neighbourhood $O(z_0)$ of z_0 such that $z \in O(z_0)$ implies

(9) $$(f_i)_*(z) > (f_i)_*(z_0) - \frac{\delta}{3},$$

where $m < i \leqslant n$ (see (6)). Now from (7) it follows that for every $z \in O(z_0)$ there is some n_z with $m + 1 \leqslant n_z \leqslant n$ such that

$$f_*(z) > (f_{n_z})_*(z) - \frac{\delta}{3} > (f_{n_z})_*(z_0) - \frac{2\delta}{3} > f_*(z) - \delta$$

(here the second inequality follows from (9), and the third from (8)). Hence $f_*(z) \geqslant f_*(z_0)$ for every $z \in O(z_0)$ and consequently f_* is lower semi continuous. A similar argument shows that f^* is upper semi-continuous and hence by Lemma 5, $f \in \mathscr{D}$, q. e. d.

The next theorem is more general than Theorem 3. It gives a sufficient condition for the limit f of a sequence $\{f_n\}_{n=1}^{\infty}$ of functions to be in $\mathscr{D}$, where f_n are arbitrary functions $f_n : R_0 \to R_0$ such that $\tilde{f}_n \to \tilde{f}$. First we prove the following lemma:

Lemma 9. *Let T be a first countable topological space. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : T \to R$ which converges pointwise to a function f. Then f is lower (upper) semi-continuous if and only if for every $a \in T$ and every $\varepsilon > 0$ there is a neighbourhood $O(a)$ of a such that for every $z \in O(a)$ and every k there is some m with*

$$f_{k+m}(z) > f(a) - \varepsilon \qquad (\text{resp. } f_{k+m}(z) < f(a) + \varepsilon);$$

in symbols

(10) $$\forall_{a}\ \forall_{\varepsilon>0}\ \exists_{O(a)}\ \forall_{z\in O(a)}\ \forall_{k}\ \exists_{m}\ f_{k+m}(z) > f(a) - \varepsilon \quad (\textit{resp. } f_{k+m}(z) < f(a) + \varepsilon).$$

Proof: Because of the symmetry it suffices to prove the Lemma for lower semi-continuous functions. Thus assume the condition (10) to be satisfied. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence converging in T to a. We can assume $z_n \in O(a)$, for every n. Since f_n converge to f there is a k_1 such that $(fz_1) > f_k(z_1) - \varepsilon$, for $k > k_1$. In general, let k_n be a positive integer such that for every $k > k_n$, $f(z_n) > f_k(z_n) - \varepsilon$. From (10) it follows that there is a sequence $\{m_i\}_{i=1}^{\infty}$ of positive integers such that $f_{k_n+m_n}(z_n) > f(a) - \varepsilon$, for every n. Hence

$$f(z_n) > f_{k_n+m_n}(z_n) - \varepsilon > f(a) - 2\varepsilon$$

and hence

$$\liminf_{n\to\infty} f(z_n) \geqslant f(a) - 2\varepsilon;$$

thus $\liminf_{n\to\infty} f(z_n) \geqslant f(a)$ and consequently (see (6)) f is lower semi-continuous.

Now assume that a sequence $\{f_n\}_{n=1}^{\infty}$ converges to a lower semi-continuous function f and that contrary to what we wish to show the condition (10) is not satisfied. Then

$$\exists_{a}\ \exists_{\varepsilon>0}\ \forall_{O(a)}\ \exists_{z\in O(a)}\ \exists_{k}\ \forall_{m}\ f_{k+m}(z) \leqslant f(a) - \varepsilon.$$

Hence in every neighbourhood of a there is a point z such that, for every m, $f_{k+m}(z) \leqslant f(a) - \varepsilon$, so that $\lim_{m\to\infty} f_{k+m}(z) = f(z) \leqslant f(a) - \varepsilon$. But in this case f cannot be lower semi-continuous (see (6)) in a. The contradiction finishes the proof of the Lemma.

Now we are able to prove the following.

Theorem 4. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : R_0 \to R_0$ converginy pointwise to a function f such that $\tilde{f}_n$ converges to $\tilde{f}$. Then f is in $\mathscr{D}$ if and only if for every $a \in S$, and $\varepsilon < 0$, there is a neighbourhood $O(a)$ of a such that for every $z \in O(a)$,*

$$\tilde{f}(z) \subset O_\varepsilon(\tilde{f}(a)).$$

Proof: S is a first countable topological space (see the section 2 above) hence Lemma 9 can be applied. Replace the functions f_{k+m}, f, in (10) by $(f_{k+m})_*$, f_*, resp. $(f_{k+m})^*$, f^*, to obtain the condition

$$\forall_{a}\ \forall_{\varepsilon>0}\ \exists_{O(a)}\ \forall_{z\in O(a)}\ \forall_{k}\ \exists_{m}\ \tilde{f}_{k+m}(z) = \langle (f_{k+m})_*(z), (f_{k+m})^*(z)\rangle \subset O_{\varepsilon/2}(\tilde{f}(a));$$

From this the Theorem follows.

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Received March 5, 1970

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