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Periodic Series

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MATEMATICKÝ ČASOPIS

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PERIODIC SERIES

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Some years ago I was concerned with groups expressible as unions of disjoint subsemigroups (see [1] and [2]), and I found it useful to introduce the *periodic series*

$$1 = \pi_0(G) \subseteq \pi_1(G) \subseteq \ldots \subseteq \pi_\lambda(G) \subseteq \ldots$$

of an arbitrary group G, defined as follows. If λ is a limit ordinal, $\pi_\lambda(G)$ is the union of all $\pi_\mu(G)$ with $\mu < \lambda$. If λ has an immediate predecessor μ, then $\pi_\lambda(G)$ is the subgroup of G generated by the elements having finite order modulo $\pi_\mu(G)$. The limit $\pi_*(G)$ of the periodic series is the *peak* of G; evidently, G has a non-trivial homomorphic image without non-trivial elements of finite order if and only if $\pi_*(G) \neq G$. If G is expressed as a union of disjoint subsemigroups, then the peak lies in the subsemigroup containing the identity element.

One is led to inquire as to possible lengths of periodic series. They can not be arbitrarily long, as our first and very trivial result shows.

Lemma 1. *For any group G, $\pi_\omega(G) = \pi_{\omega+1}(G)$.*

Proof. Here ω is the first infinite ordinal. Let x have finite order modulo $\pi_\omega(G)$. Then x has finite order modulo $\pi_n(G)$ for some $n < \omega$, so that $x \in \pi_{n+1}(G)$, and $x \in \pi_\omega(G)$. This proves the lemma.

Now one asks: given an ordinal $\alpha \leqslant \omega$, is there a group whose periodic series has length exactly α? The answer is yes, as we shall see presently. *We shall make a group G_α whose periodic series has length precisely α and such that $\pi_\alpha(G_\alpha) = G_\alpha$.* First some preliminaries.

Lemma 2. *For any integers $i, j \geqslant 0$, $\pi_i(G/\pi_j(G)) = \pi_{i+j}(G)/\pi_j(G)$.*

Proof. This follows by an easy induction on i.

We also omit the proof of our third lemma; here an easy induction on n gives the answer.

Lemma 3. *If G is the restricted direct product of subgroups G_i, then for each finite n, $\pi_n(G)$ is the restricted direct product of the $\pi_n(G_i)$.*

Our last preliminary is a well-known result on generalized free products.

The essay [3] is a convenient reference.

Let G be the generalized free product of two groups A and B with amalgamation. Then the only elements of finite order in G are conjugates of elements of finite order in A or B.

Using these results we construct a sequence

$$G_1, G_2, \ldots, G_n, \ldots$$

of groups with the desired properties. First, G_1 is an infinite dihedral group generated by elements a_1 and b_1 of order 2. For each $n \geqslant 1$, G_{n+1} is to be a generalized free product of G_n and the group B_{n+1} generated by two elements a_{n+1}, b_{n+1} with the single defining relation $a_{n+1}^2 = b_{n+1}^2$. As a generalized free product of infinite cyclic groups, B_{n+1} has no elements of finite order except the identity. Precisely, G_{n+1} is to be generated by $2(n+1)$ elements a_1, $b_1, \ldots, a_{n+1}, b_{n+1}$ subject to the defining relations

$$a_1^2 = b_1^2 = 1, \; a_{i+1}^2 = b_{i+1}^2 = a_i b_i, \; i = 1, 2, \ldots, n.$$

It is clear from the defining relations that G_{n+1} is the generalized free product of G_n and B_{n+1}, amalgamating the infinite cyclic subgroup generated by $a_n b_n$ in G_n with that generated by a_{n+1}^2 in B_{n+1}. Moreover, $\pi_{n+1}(G_{n+1}) = G_{n+1}$ and, by the above-quoted result on generalized free products, $\pi_1(G_{n+1})$ is the normal closure of G_1 in G_{n+1}. This means that $G_{n+1}/\pi_1(G_{n+1})$ is the group obtained from G_{n+1} by putting $a_1 = b_1 = 1$; and a glance at the defining relations shows that this factor-group is isomorphic with G_n.

From Lemma 2 it now follows immediately that, for $i \leqslant n+1$, $\pi_i(G_{n+1})$ is the normal closure of G_i in G_{n+1}. In any case

$$1 = \pi_0(G_{n+1}) \subset \pi_1(G_{n+1}) \subset \ldots \subset \pi_{n+1}(G_{n+1}) = G_{n+1},$$

as we wanted.

Lastly, let G stand for the direct product of all the G_i. Then, by Lemma 3, $\pi_n(G) \neq G$ for all n, and $\pi_\omega(G) = G$.

REFERENCES

[1] Wiegold J., *Semigroup coverings of groups*, Mat.-fyz. časop. *11* (1961), 3—13.

[2] Wiegold J., *Semigroup coverings of groups II*, Mat.-fyz. časop. *12* (1962), 217—223.

[3] Neumann B. H., *An essay on free products of groups with amalgamations*, Phil. Trans. Roy. Soc. London Ser. A *246* (1954), 503—554.

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