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ON CERTAIN THEOREMS OF BERRY AND A LIMIT THEOREM OF FELLER

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The purpose of this paper is to improve the statements of certain theorems due to Berry (Section 1) and a limit theorem due to Feller (Section 2).

1.

Let X_k, $k = 1,2, \dots, n$ be independent random variables, let $F_k(x)$, α_k, $\sigma_k^2 > 0$, $\mu_{3k} < \infty$, $k = 1,2, \dots n$ be their distribution functions, mean values, variances and third absolute central moments, respectively.

Let

$$X = \sum_{k=1}^{n} X_k; \tag{1}$$

denote $F(x)$, $\alpha = \sum_{k=1}^{n} \alpha_k$, $\sigma^2 = \sum_{k=1}^{n} \sigma_k^2$, the distribution function, mean value and variance of X, respectively.

We define

$$\tilde{\varepsilon} = \frac{\sum_{k=1}^{n} \mu_{3k}}{\sigma^3}, \tag{2}$$

$$M = \sup_{-\infty < x < +\infty} \left| F(x) - \Phi\left(\frac{x - \alpha}{\sigma}\right) \right|, \tag{3}$$

where

$$\Phi(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt, \ x \in (-\infty, \infty). \tag{4}$$

Further we define the „moment — ratio" λ_k as follows:

$$\lambda_k = \frac{\mu_{3k}}{\sigma_k^2}, \quad k = 1, 2, \dots, n \tag{5}$$

and put

$$\varepsilon = \frac{1}{\sigma} \max_{1 \leq k \leq n} \lambda_k .$$

It can be easily shown that $\tilde{\varepsilon} \leqslant \varepsilon$.

A. C. Berry [1] has shown that there exists an absolute constant C (independent of the $F_k's$) such that

$$M \leqslant C\varepsilon .$$

He also gave an upper bound for it:

$$C \leqslant 1{,}88;$$

the proof of the last inequality contains however, an error, which was corrected by K. Takano [3], who obtained thus only the estimate

$$C < 2{,}031 .$$

Berry's method of proof has been essentially refined by B. M. Zolotarev [4], who obtained an estimate

$$C < 1{,}322, \tag{6}$$

even for the inequality $M \leqslant C\tilde{\varepsilon}$.

In the mentioned paper Takano pointed out that also other theorems in [1] will have to be corrected. Theorems 3, 4, 5 in [1] hold because of our relation (6), but, as we shall show below, they can be improved.

Let us now assume that $F_k(x)$, $k = 1, 2, \ldots, n$ are the distribution functions of independent random variables X_k, $k = 1, 2, \ldots, n$, $F(x)$ the distribution function of the sum $X = \sum_{k=1}^{n} X_k$ and the function $\Phi(x)$ defined as in (4). Let a_k $(k = 1, 2, \ldots, n)$, $b > 0$ be real numbers and $a = \sum_{k-1}^{n} a_k$. Let

$$\bar{M} = \sup_{-\infty < x < +\infty} \left| F(x) - \Phi\left(\frac{x-a}{b}\right) \right| . \tag{7}$$

For a given $\bar{\varkappa} > 0$, we define the following quantities

$$\bar{\varkappa}_0 = \sum_{k=1}^{n} \mathbf{P}\left\{|X_k - a_k| > \bar{\varkappa} b\right\}, \tag{8}$$

$$\bar{\varkappa}_1 = \frac{1}{b} \sum_{k=1}^{n} \left| \int_{a_k - \bar{\varkappa} b}^{a_k + \bar{\varkappa} b} (x - a_k)\, dF_k(x) \right| , \tag{9}$$

(10) $$\bar{\varkappa}_2 = \left| 1 - \frac{1}{b^2} \sum_{k=1}^{n} \int_{a_k - \bar{\varkappa} b}^{a_k + \bar{\varkappa} b} (x - a_k)^2 dF_k(x) \right| .$$

Theorem 1. *If* $\bar{\varkappa}_0 \leqslant \bar{\varkappa}$, $\bar{\varkappa}_1 \leqslant \bar{\varkappa}$, $\bar{\varkappa}_2 \leqslant \bar{\varkappa}$, *then*

(11) $$\bar{M} < 4{,}647\bar{\varkappa} .$$

Proof. We shall use the correct part of Berry's Theorem 3 in [1], which states

(12) $$\bar{M} \leqslant \frac{C(\bar{\varkappa} + \bar{\varkappa}_1)}{(1 - \bar{\varkappa}_1^2 - \bar{\varkappa}_2)^{\frac{1}{2}}} + \bar{\varkappa}_0 + \frac{\bar{\varkappa}_1}{(2\pi)^{\frac{1}{2}}} + \frac{1}{(2\pi e)^{\frac{1}{2}}} \log \frac{1}{(1 - \bar{\varkappa}_1^2 - \bar{\varkappa}_2)^{\frac{1}{2}}}$$

Using further the inequalities $C < 1{,}322$, $\bar{\varkappa}_k \leqslant \bar{\varkappa}$, $k = 0, 1, 2$ we get

(13) $$\bar{M} < \bar{\varkappa} g(\bar{\varkappa}) ,$$

where

(14) $$g(\bar{\varkappa}) = \frac{2{,}644}{(1 - \bar{\varkappa}^2 - \bar{\varkappa})^{\frac{1}{2}}} + 1 + \frac{1}{(2\pi)^{\frac{1}{2}}} + \frac{1}{\bar{\varkappa}(2\pi e)^{\frac{1}{2}}} \cdot \log \frac{1}{(1 - \bar{\varkappa}^2 - \bar{\varkappa})^{\frac{1}{2}}} .$$

Since $\bar{M} \leqslant 1$, the theorem is trivial for $\bar{\varkappa} > \frac{1}{4{,}647}$; assume therefore that $\bar{\varkappa} \in \left(0, \frac{1}{4{,}647}\right\rangle$. In this interval, $g(\bar{\varkappa})$ is an increasing function of $\bar{\varkappa}$ and $g(\bar{\varkappa}) < < 4{,}647$ which proves the theorem.

From now on let us assume that the mean values α_k and the variances $\sigma_k^2 > 0$ $(k = 1, 2, \ldots, n)$ of the random variables are finite. Let us use the symbols $\alpha = \sum_{k=1}^{n} \alpha_k$ and $\sigma^2 = \sum_{k=1}^{n} \sigma_k^2$ for the mean value and the variance of the sum $X = \sum_{k=1}^{n} X_k$. In this case $\bar{M} = M$, where $a_k = \alpha_k$, $k = 1, 2, \ldots, n$ and $b = \sigma$.

For quantities $\varkappa$, $\varkappa_0$, $\varkappa_1$, $\varkappa_2$ corresponding to $\bar{\varkappa}$, $\bar{\varkappa}_0$, $\bar{\varkappa}_1$, $\bar{\varkappa}_2$ in the preceding case, we can write

(15) $$\varkappa_0 = \sum_{k=1}^{n} \mathbf{P}\{|X_k - \alpha_k| > \varkappa\sigma\} ,$$

(16) $$\varkappa_1 = \frac{1}{\sigma} \sum_{k=1}^{n} \left| \left(\int_{-\infty}^{\alpha_k - \varkappa\sigma} + \int_{\alpha_k + \varkappa\sigma}^{\infty} \right) (x - \alpha_k)\, dF_k(x) \right| ,$$

$$(17) \qquad \varkappa_2 = \frac{1}{\sigma^2}\sum_{k-1}^{n}\left(\int_{-\infty}^{\alpha_k-\varkappa\sigma} + \int_{\alpha_k+\varkappa\sigma}^{\infty}\right)(x-\alpha_k)^2 dF_k(x).$$

We prove the following

Theorem 2. *If* $\varkappa_2 \leqslant \varkappa^3$, *then*

$$(18) \qquad M < 3{,}188\varkappa.$$

Proof. To get a non-trivial case, let us assume that $\varkappa \in \left(0, \frac{1}{3{,}188}\right\rangle$; we shall again use the correct part of Berry's Theorem 3 in [1] and the inequalities $\varkappa_0 \leqslant \varkappa$, $\varkappa_1 \leqslant \varkappa^2$, $\varkappa_2 \leqslant \varkappa^3$, (where the first and the second inequalities follow from (15), (16), and from the third), we get

$$(19) \qquad M < \varkappa g_1(\varkappa),$$

where

$$(20) \quad g_1(\varkappa) = \frac{1{,}322(1+\varkappa)}{(1-\varkappa^4-\varkappa^3)^{\frac{1}{2}}} + 1 + \frac{1}{(2\pi)^{\frac{1}{2}}} + \frac{1}{\varkappa(2\pi e)^{\frac{1}{2}}} \cdot \log \frac{1}{(1-\varkappa^4-\varkappa^3)^{\frac{1}{2}}}.$$

In the interval $\left(0, \frac{1}{3{,}188}\right\rangle$, $g_1(\varkappa)$ is an increasing function of $\varkappa$ and $g_1(\varkappa) < 3{,}188$ which proves the theorem.

A simple consequence of Theorem 2 is

Theorem 3. *Let*

$$(21) \qquad \mu_{s,k} = \mathbf{E}(|X_k - \alpha_k|^s), \qquad k = 1, 2, \ldots, n$$

be finite for $s > 2$ (*not necessarily integer*). *Then*

$$(22) \qquad M < 3{,}188(\varepsilon^*)^{\frac{1}{s+1}}, \qquad \textit{where}$$

$$(23) \qquad \varepsilon^* = \frac{1}{\sigma^s}\sum_{k-1}^{n}\mu_{s,k}.$$

Proof. We put $\varkappa = (\varepsilon^*)^{\frac{1}{s+1}}$. Further, we have

$$(24) \qquad \mu_{s,k} \geqslant \int_{|x-\alpha_k|\geq\varkappa\sigma} |x-\alpha_k|^s dF_k(x) \geqslant \varkappa^{s-2}\frac{\sigma^s}{\sigma^2}\int_{|x-\alpha_k|\geq\varkappa\sigma}|x-\alpha_k|^2 dF_k(x)$$

Summing over k and dividing by σ^s gives

$$\varepsilon^* \geqslant \varkappa^{s\ 2}\varkappa_2, \quad \text{i. e.} \quad \varkappa^3 \geqslant \varkappa_2. \tag{25}$$

Remark. Berry in [1] (Theorems 3, 4, 5) gives the constants 5,8 and 3,6 instead of our 4,647 and 3,188 respectively.

2.

Feller's Theorem 1 in [2], which is a generalization of Cramer's limit theorem can be improved using the results of the preceding section, and also by improving some estimates used in Feller's proof. As we shall demonstrate in the sequel, the technique of the proof itself remains unchanged.

Let $F_k(x)$, $k = 1, 2, \ldots, n$ be distribution functions of independent random variables X_k, $k = 1, 2, \ldots, n$. Suppose that

$$\mathbf{E}(X_k) = 0, \quad \mathbf{E}(X_k^2) = \sigma_k^2, \quad 0 < \sigma_k^2 < +\infty, \tag{1}$$
$$k = 1, 2, \ldots, n.$$

Further, let

$$X = \sum_{k=1}^{n} X_k. \tag{2}$$

The mean value of X is

$$\mathbf{E}(X) = 0 \tag{3}$$

and its variance is

$$\sigma^2 = \sum_{k=1}^{n} \sigma_k^2. \tag{4}$$

From now on we shall suppose that, in addition to (1) the random variables X_k satisfy for some $\lambda > 0$ the condition

$$|X_k| < \lambda\sigma \text{ for } k = 1, 2, \ldots, n. \tag{5}$$

Further, let $F(x)$ be the distribution function of the random variable X.

With the help of a suitable transformation (cf. [2]) we can, if we choose a suitable real parameter $h > 0$ transform the sequence $\{F_k(x)\}_{k=1}^{n}$ of distribution functions into another sequence of distribution functions $\{\overline{F}_k(x)\}_{k=1}^{n}$.

The transformation is defined by

$$V_k = \int_{-\infty}^{\infty} e^{hx} dF_k(x) \tag{6}$$

and

$$(7) \qquad \bar{F}_k(x) = \frac{1}{V_k} \int_{-\infty}^{x} e^{hy} dF_k(y).$$

Let $\bar{X}_k$ $(k = 1, 2, \ldots, n)$ be the independent random variables corresponding to these distribution functions $\bar{F}_k(x)$.

The mean value of the random variable $\bar{X}_k$ will be denoted by $\bar{\alpha}_k$ and its variance by $\bar{\sigma}_k^2$.

Let $\bar{F}(x)$ be the distribution function of the random variable

$$(8) \qquad \bar{X} = \sum_{k=1}^{n} \bar{X}_k,$$

let

$$(9) \qquad \bar{\alpha} = \sum_{k=1}^{n} \bar{\alpha}_k, \quad \bar{\sigma}^2 = \sum_{k=1}^{n} \bar{\sigma}_k^2$$

be its mean value and variance.

Using (7) (see [2]) it is easy to prove the following.

Lemma 1. *We have*

$$(10) \qquad 1 - F(x) = V_1 V_2 \ldots V_n \int_{(x-\bar{\alpha})/\bar{\sigma}}^{\infty} \bar{e}^{h(\bar{\alpha}+y\bar{\sigma})} d\bar{F}(\bar{\alpha} + y\bar{\sigma}).$$

Corollary. *Let*

$$(11) \qquad V = \log \prod_{k=1}^{n} V_k.$$

For $x = \bar{\alpha}$ *the relation* (10) *implies*

$$(12) \qquad 1 - F(\bar{\alpha}) = e^{V-h\bar{\alpha}} \int_{0}^{\infty} e^{-hy\bar{\sigma}} d\bar{F}(\bar{\alpha} + y\bar{\sigma}).$$

The semiinvariants $\gamma_{k,\nu}$ $(k = 1, 2, \ldots, n;\ \nu = 2, 3, \ldots)$ of the functions $F_k(x)$ are defined by the relation

$$(13) \qquad \log \int_{-\infty}^{\infty} e^{hy} dF_k(y) = \sum_{\nu=2}^{\infty} \gamma_{k,\nu} \frac{h^\nu}{\nu!}.$$

We put

$$(14) \qquad \Gamma_\nu = \sum_{k-1}^{n} \gamma_{k,\nu}.$$

From (13) and (6) we get

$$(15) \qquad \log V_k = \sum_{\nu=2}^{\infty} \gamma_{k,\nu} \frac{h^\nu}{\nu!}.$$

From the first derivative of (15) with respect to h we get

$$\bar{\alpha}_k = \sum_{\nu=2}^{\infty} \gamma_{k,\nu} \frac{h^{\nu-1}}{(\nu-1)!}, \tag{16}$$

and the second derivative of (15) with respect to h gives

$$\bar{\sigma}_k^2 = \sum_{\nu=2}^{\infty} \gamma_{k,\nu} \frac{h^{\nu-2}}{(\nu-2)!}. \tag{17}$$

Formulae (11), (14), (15), (16), (17) and (9) imply

$$V = \sum_{\nu=2}^{\infty} \Gamma_\nu \frac{h^\nu}{\nu!}, \tag{18}$$

$$\bar{\alpha} = \sum_{\nu=2}^{\infty} \Gamma_\nu \frac{h^{\nu-1}}{(\nu-1)!}. \tag{19}$$

$$\bar{\sigma}^2 = \sum_{\nu=2}^{\infty} \Gamma_\nu \frac{h^{\nu-2}}{(\nu-2)!}. \tag{20}$$

The following lemma is due to Feller (cf. [2]):

Lemma 2. *Let $\lambda > 0$ be the constant in formula* (5). *Then for $\nu \geqslant 3$, $k = 1, 2, \ldots, n$ the following estimates hold*:

$$|\gamma_{k,\nu}| \leqslant \frac{(\nu-2)!}{2}\, \sigma_k^2 (2\lambda\sigma)^{\nu-2}, \tag{21}$$

$$|\Gamma_\nu| \leqslant \frac{(\nu-2)!}{2}\, \sigma^2 (2\lambda\sigma)^{\nu-2}. \tag{22}$$

In the next lemma, the results depend not only on the interval containing $\lambda\sigma h$, but also on the estimate (18) of the preceding section.

Lemma 3. *Let*

$$0 < \lambda\sigma h < \frac{13}{144}. \tag{23}$$

Then

$$(24)\qquad |\bar{\alpha}_k| < \frac{1079}{11328}\lambda\sigma, \quad k = 1, 2, \ldots, n$$

and

$$(25)\qquad \left|\frac{\bar{\sigma}^2}{\sigma^2} - 1\right| < \frac{13}{118}.$$

Further,

$$(26)\qquad 1 - F(\bar{\alpha}) = e^{V-h\bar{\alpha}}\{e^{\frac{1}{2}(h\bar{\sigma})^2}(1 - \Phi(h\bar{\sigma})) + \Theta_1\lambda\},$$

where

$$(27)\qquad |\Theta_1| < 7{,}403016.$$

Proof. Using (16) and (21) we get

$$(28)\qquad |\bar{\alpha}_k| \leqslant |\gamma_{k,2}|\, h + \sum_{\nu=3}^{\infty} |\gamma_{k,\nu}| \frac{h^{\nu-1}}{(\nu-1)!} \leqslant \sigma_k^2 h \left\{1 + \frac{1}{2}\sum_{\nu=3}^{\infty}\frac{1}{\nu-1}(2\lambda\sigma h)^{\nu-2}\right\}$$

$$< \lambda^2\sigma^2 h\left\{1 + \frac{1}{2}\frac{\lambda\sigma h}{1 - 2\lambda\sigma h}\right\}.$$

Since $0 < \lambda\sigma h < \frac{13}{144}$ by assumption, formula (28) implies the relation (24).

Further,

$$(29)\qquad \bar{\sigma}^2 = \Gamma_2 + \sum_{\nu=3}^{\infty}\Gamma_\nu \frac{h^{\nu-2}}{(\nu-2)!} = \sigma^2 + \sum_{\nu=3}^{\infty}\Gamma_\nu \frac{h^{\nu-2}}{(\nu-2)!}$$

according to (13), (14) and (20).

Therefore

$$(30)\qquad |\bar{\sigma}^2 - \sigma^2| \leqslant \sum_{\nu=3}^{\infty}|\Gamma_\nu|\frac{h^{\nu-2}}{(\nu-2)!} \leqslant \sigma^2\frac{\lambda\sigma h}{1 - 2\lambda\sigma h}.$$

For $0 < \lambda\sigma h < \frac{13}{144}$, $\lambda\sigma > 0$, we get

$$(31)\qquad |\bar{\sigma}^2 - \sigma^2| < \frac{13}{118}\sigma^2$$

and therefrom directly follows (25). Using (5) and (24) we get for $k = 1, 2, \ldots, n$

$$(32)\qquad |X_k - \bar{\alpha}_k| < \left(1 + \frac{1079}{11328}\right)\lambda\sigma.$$

Using (32) and (25) we have for $k = 1, 2, \ldots, n$

(33) $$|X_k - \bar{\alpha}_k| < \Delta\, \lambda\bar{\sigma}, \text{ where}$$

(34) $$\Delta = \frac{12407}{11328}\sqrt{\frac{118}{105}} < 1{,}161075.$$

Therefore for $k = 1, 2, \ldots, n$ we have

(35) $$\int\limits_{|x|>\Delta\lambda\bar{\sigma}} d\bar{F}_k(x + \bar{\alpha}_k) \leqslant \frac{1}{V_k} \int\limits_{|x|>\lambda\sigma} e^{hx} dF_k(x) = 0.$$

In Theorem 3 of the preceding section, let us put $\varkappa = \Delta\lambda$. Then from (33) $\varkappa_0 = \varkappa_1 - \varkappa_2 - 0$ and the conditions of the theorem are satisfied. In this way we have proved that

(36) $$\bar{F}(\bar{\alpha} + x\bar{\sigma}) - \Phi(x) = R(x),$$

where

(37) $$|R(x)| < 3{,}701508\lambda.$$

Using (36) and (12) we have

(38) $$1 - F(\bar{\alpha}) = e^{V-h\bar{\alpha}}\{(2\pi)^{-\frac{1}{2}} \int\limits_0^\infty e^{-hy\bar{\sigma} - \frac{1}{2}y^2}\, dy + \int\limits_0^\infty e^{-hy\bar{\sigma}} dR(y)\}.$$

The following estimate of the absolute value of the last integral in (38) follows from (37):

(39) $$\left|\int\limits_0^\infty e^{-hy\sigma} dR(y)\right| \leqslant |R(0)| + h\bar{\sigma}\left|\int\limits_0^\infty e^{-hy\sigma} R(y) dy\right| < 7{,}403016\lambda.$$

Further,

(40) $$(2\pi)^{-\frac{1}{2}} \int\limits_0^\infty e^{-hy\sigma - \frac{1}{2}y^2}\, dy = e^{\frac{1}{2}(h\bar{\sigma})^2} (1 - \Phi(h\bar{\sigma})).$$

From (40), (39) and (38) we obtain the remaining proofs of (26) and (27).

Now let $h = h(x) > 0$ be a function of x defined with the help of the inverse function x:

(41) $$x = \frac{1}{\sigma} \sum_{\nu\ 2}^{\infty} \Gamma_\nu \frac{h^{\nu-1}}{(\nu - 1)!} \equiv \frac{\bar{\alpha}}{\sigma}.$$

A formal calculation of inverse series to (41) gives

$$h = \frac{x}{\sigma} - \frac{x^2}{2\sigma^4} \Gamma_3 + \dots . \tag{42}$$

Using (22) we see (Cauchy's principle) that the inversion of the series (41) is possible for at least all those values of x for which we can invert the series

$$h = \frac{x}{\sigma} + \frac{1}{4\sigma} \sum_{\nu=3}^{\infty} (2\lambda)^{\nu-2} (\sigma h)^{\nu-1}. \tag{43}$$

The inversion of (43) is possible for

$$0 < h < \frac{1}{2\lambda\sigma}, \tag{44}$$

which corresponds to the interval (cf. [2])

$$0 < \lambda x < (3 - 5^{\frac{1}{2}})/4. \tag{44'}$$

Feller (in [2]) proved for these x and h the following estimate using (43)

$$h - x \prec \frac{x}{12} \frac{6\lambda x}{1 - 6\lambda x}, \tag{45}$$

where $\prec$ denotes the fact that the expression on the right hand side majorizes the expression on the left.

Remark. Let $0 < \lambda x < \frac{1}{12}$; then (from (45)] we get $0 < \lambda\sigma h < \frac{13}{144}$, which is the hypothesis of Lemma 3.

Feller defines the function

$$Q(x) = \sum_{\nu\ 1}^{\infty} q_\nu x^\nu \tag{46}$$

by the formula

$$\frac{x^2}{2} (1 + Q(x)) = \sum_{\nu-2}^{\infty} \Gamma_\nu \frac{\nu - 1}{\nu!} h^\nu, \tag{47}$$

where $h = h(x) > 0$ is defined through its inverse function x in (41), and he further defines the function

$$Q^*(x) = \sum_{\nu=1}^{\infty} q_\nu^* x^\nu \tag{48}$$

by the formula

$$\frac{x^2}{2}(1 + Q^*(x)) = \frac{(\sigma^* h^*)^2}{2}\left\{1 + \frac{1}{3}\sum_{\nu=3}^{\infty}(2\lambda\sigma^* h^*)^{\nu-2}\right\}, \tag{49}$$

where

$$\sigma^* h^* = x + \frac{x}{12}\frac{6\lambda x}{1 - 6\lambda x}. \tag{50}$$

Evidently the interval of those values of x for which the series (49), (50) are convergent lies within the interval of convergence of the corresponding series (47), (45).

Further, according to (18) and (19)

$$\sum_{\nu-2}^{\infty} \Gamma_\nu \frac{\nu - 1}{\nu!} h^\nu = h\bar{\alpha} - V, \tag{51}$$

so that from (47) and (51) we get

$$\frac{x^2}{2}(1 + Q(x)) = h\bar{\alpha} - V, \tag{52}$$

the series (51) and (52) being convergent if (44) is satisfied, owing to the validity of (22), at least on the interval (44').

The following lemma now holds:

Lemma 4. *Let* $0 < \lambda x < \frac{1}{12}$. *Then series* (49) *and* (50) *are majorants for the series* (47) *and* (54), *respectively.*

Moreover, the coefficients q_ν, $\nu = 1, 2, \ldots$, *defined through* (46) *and* (47), *depend only on* $\nu + 2$ *first moments of* X_k, $k = 1, 2, \ldots, n$, *and they satisfy*

$$|q_1| \leqslant \frac{1}{3}\lambda \tag{53}$$

and

$$q_\nu| < \frac{1}{8}(12\lambda)^\nu, \quad \textit{for} \quad \nu = 2, 3, \ldots. \tag{53'}$$

Proof. The coefficient of x^s — say c_s — in the inverse series $h = h(x)$ depends only on coefficients of $h, h^2, \ldots, h^s$ in the original series (47), i. e. only on the quantities $\Gamma_2, \ldots, \Gamma_{s+1}$; inserting $h = h(x)$ into the series (41) and grouping the terms with the same power, we find out that the coefficient of x^s in this series depends on $c_1, c_2, \ldots, c_{s-1}$, i. e. on $\Gamma_2, \ldots, \Gamma_s$. By comparing we get that q_s depends on $\Gamma_2, \ldots, \Gamma_{s+2}$; these quantities can be expressed by moments of X_k, $k = 1, 2, \ldots, n$, of order at most $s+2$.

The fact of majorizing is evident from (22); therefore

$$|q_\nu| \leqslant q_\nu^*, \qquad \nu = 1, 2, \ldots . \tag{54}$$

Simple computation shows that

$$Q^*\left(\frac{1}{12\lambda}\right) < 0{,}259809.$$

Further, $q_1^* = \dfrac{5}{3}\lambda$. Using this, we obtain for q_ν^*, $\nu = 2, 3, \ldots$, the estimate

$$q_\nu^* < 0{,}120920\,(12\lambda)^\nu < \tfrac{1}{8}\,(12\lambda)^\nu,$$

which implies (53′). Using (47), (42), (46), (47) and (22) we get the following estimate for q_1:

$$|q_1| = \frac{1}{3\sigma^3}\,|\Gamma_3| \leqslant \frac{1}{3}\,\lambda. \tag{55}$$

Lemma 5. *Let $h > 0$, $\bar\sigma$ and x satisfy the relations* (20) *and* (41) *and let* $0 < \lambda\sigma h < \dfrac{1}{3}$. *Then*

$$|h\bar\sigma - x| \leqslant \lambda^2(\sigma h)^3\left\{\frac{1}{1-2\lambda\sigma h} + \frac{1}{8(1-2\lambda\sigma h)\,(1-3\lambda\sigma h)}\right\}. \tag{56}$$

Proof. From (41) we have

$$x = \sigma h\left(1 + \frac{1}{\sigma^2}\sum_{\nu=3}^{\infty}\Gamma_\nu\frac{h^{\nu-2}}{(\nu-1)!}\right) \tag{57}$$

Furthermore from (20)

$$h\bar\sigma = \sigma h\left(1 + \frac{1}{\sigma^2}\sum_{\nu=3}^{\infty}\Gamma_\nu\frac{h^{\nu-2}}{(\nu-2)!}\right)^{\frac{1}{2}} \tag{58}$$

Using (57) and (58) we get

$$h\bar\sigma - x = \sigma h\left\{\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}\left(\frac{1}{\sigma^2}\sum_{\nu=3}^{\infty}\frac{\Gamma_\nu}{(\nu-2)!}\,h^{\nu-2}\right)^k - \frac{1}{\sigma^2}\sum_{\nu-3}^{\infty}\Gamma_\nu\frac{h^{\nu-2}}{(\nu-1)!}\right\} = \tag{59}$$

$$- \sigma h \left\{ \frac{1}{2} \frac{1}{\sigma^2} \sum_{\nu=3}^{\infty} \frac{\Gamma_\nu}{(\nu-2)!} h^{\nu-2} + \sum_{k=2}^{\infty} \binom{\frac{1}{2}}{k} \left(\frac{1}{\sigma^2} \sum_{\nu=3}^{\infty} \frac{\Gamma_\nu}{(\nu-2)!} h^{\nu-2} \right)^k - \right.$$

$$\left. - \frac{1}{\sigma^2} \sum_{\nu=3}^{\infty} \Gamma_\nu \frac{h^{\nu-2}}{(\nu-2)!} \right\} = \sigma h \left\{ \frac{1}{\sigma^2} \sum_{\nu=4}^{\infty} \frac{\Gamma_\nu}{(\nu-2)!} \left(\frac{1}{2} - \frac{1}{\nu-1} \right) h^{\nu-2} + \right.$$

$$\left. + \sum_{k=2}^{\infty} \binom{\frac{1}{2}}{k} \left(\frac{1}{\sigma^2} \sum_{\nu=3}^{\infty} \frac{\Gamma_\nu}{(\nu-2)!} h^{\nu-2} \right)^k \right\}.$$

Therefore, using (22), we get

$$(60) \quad \left| \frac{1}{\sigma^2} \sum_{\nu=4}^{\infty} \frac{\Gamma_\nu}{(\nu-2)!} h^{\nu-2} \left(\frac{1}{2} - \frac{1}{\nu-1} \right) \right| \leqslant \frac{1}{\sigma^2} \frac{1}{2} \sum_{\nu=4}^{\infty} \frac{\sigma^2}{2} (2\lambda\sigma h)^{\nu-2} = \frac{(\lambda\sigma h)^2}{1-2\lambda\sigma h}.$$

Further, for $0 < \lambda\sigma h < \frac{1}{3}$ we have the following estimate

$$(61) \quad \left| \sum_{k=2}^{\infty} \binom{\frac{1}{2}}{k} \left(\frac{1}{\sigma^2} \sum_{\nu=3}^{\infty} \frac{\Gamma_\nu}{(\nu-2)!} h^{\nu-2} \right)^k \right| \leqslant \frac{1}{8} \sum_{k=2}^{\infty} \left(\frac{1}{\sigma^2} \sum_{\nu=3}^{\infty} \frac{|\Gamma_\nu|}{(\nu-2)!} h^{\nu-2} \right)^k \leqslant$$

$$\leqslant \frac{1}{8} \frac{(\lambda\sigma h)^2}{(1-2\lambda\sigma h)(1-3\lambda\sigma h)}.$$

The formulae (59), (60) and (61) imply

$$(62) \qquad |h\bar{\sigma} - x| \leqslant \lambda^2 (\sigma h)^3 \left\{ \frac{1}{1-2\lambda\sigma h} + \frac{1}{8(1-2\lambda\sigma h)(1-3\lambda\sigma h)} \right\},$$

which is the formula (56).

Corollary. *Let* $0 < \lambda x < \frac{1}{12}$; *then*

$$(63) \qquad 0 < \sigma h < \frac{13}{12} x.$$

For $0 < \lambda\sigma h < \frac{13}{144}$ *we have*

$$(64) \qquad |h\bar{\sigma} - x| < 1{,}817535 \lambda^2 x^3.$$

(The inequality (63) follows from (45); the inequality (64) from (56) and (63).

We define the function $f(t)$ as follows

(65) $$f(t) = e^{\frac{1}{2}t^2}\{1 - \Phi(t)\}, \qquad t > 0.$$

For $1 - \Phi(t)$, $t > 0$ we have the well-known estimates

(66) $$(2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}t^2}\left\{\frac{1}{t} - \frac{1}{t^3}\right\} < 1 - \Phi(t) < (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}t^2}\frac{1}{t}.$$

If we compute the derivatiwe of (65) and then use (66), we get the relation

(67) $$0 \leqslant -f'(t) \leqslant \frac{1}{(2\pi)^{\frac{1}{2}}t^2}, \qquad t > 0.$$

Now, it is easily seen that

(68) $$|f(h\bar{\sigma}) - f(x)| \leqslant |h\bar{\sigma} - x| \max |f'(t)| \leqslant |h\bar{\sigma} - x| \max \frac{1}{(2\pi)^{\frac{1}{2}}t^2},$$

where the maximum is taken over all t betwen $h\bar{\sigma}$ and x; this gives

(69) $$|f(h\bar{\sigma}) - f(x)| \leqslant \frac{|h\bar{\sigma} - x|}{(2\pi)^{\frac{1}{2}}(x - |h\bar{\sigma} - x|)^2},$$

when $|h\bar{\sigma} - x| < x$.

For $0 < \lambda x < \dfrac{1}{12}$ the inequality (64) implies

(70) $$|f(h\bar{\sigma}) - f(x)| < \frac{1{,}817535\lambda^2x^3}{(2\pi)^{\frac{1}{2}}(x - 1{,}817535\lambda^2x^3)^2} < 0{,}743748\lambda^2x.$$

The following lemma is easily derived from the results we have obtained:

Lemma 6. *Let* $\lambda > 0$, $0 < \lambda x < \dfrac{1}{12}$. *For* $Q(x)$, $f(x)$ *as defined in* (47) *and* (65) *respectively, we have*

(71) $$1 - F(x\sigma) = e^{-\frac{1}{2}x^2(1+Q(x))}\{f(x) + (\Theta_1 + \Theta_2)\lambda\},$$

where

$$|\Theta_1| < 7{,}403016, \quad |\Theta_2| < 0{,}743748\lambda x.$$

Proof. From (26) we get

(72) $$1 - F(\bar{\alpha}) = e^{V-h\bar{\alpha}}\{e^{\frac{1}{2}(h\bar{\sigma})^2}(1 - \Phi(h\bar{\sigma})) + \Theta_1\lambda\},$$

moreover Θ_1 satisfies (71). Using (41), (52) and (72), we get

$$1 - F(x\sigma) = e^{-\frac{1}{2}x^2(1+Q(x))}\{e^{\frac{1}{2}(h\sigma)^2}(1 - \Phi(h\bar{\sigma})) + \Theta_1\lambda\}, \tag{73}$$

the Θ_1 again satisfying (71). From relations (73), (65) and (70), the relation (71) can be directly derived, completing thus the proof.

Let us write the expressions $Q(x)$, σ, Γ_ν with the indices: $Q^{(j)}(x)$, $\sigma^{(j)}$, $\Gamma_\nu^{(j)}$, if they refer to random variables $X_1, X_2, \ldots, X_j$.

We shall now prove the following lemma:

Lemma 7. *Let* $0 < \lambda x < \frac{1}{12}$. *Then for* $1 \leqslant i < j \leqslant n$, *we have*

$$|Q^{(j)}(x) - Q^{(i)}(x)| < 1{,}256\,\frac{\sigma^{(j)2} - \sigma^{(i)2}}{\sigma^{(n)2}}. \tag{74}$$

Proof. Formula (47) implies

$$\frac{x^2}{2}(Q^{(j)}(x) - Q^{(i)}(x)) = \frac{1}{2}h^2(\sigma^{(j)2} - \sigma^{(i)2}) + \sum_{\nu=3}^{\infty}\frac{h^\nu}{\nu(\nu-2)!}\sum_{k=i+1}^{j}\gamma_{k,\nu}. \tag{75}$$

Using the estimate (22), we have

$$|Q^{(j)}(x) - Q^{(i)}(x)| \leqslant \frac{h^2}{x^2}(\sigma^{(j)2} - \sigma^{(i)2})\left(1 + \sum_{\nu=3}^{\infty}\frac{(2\lambda\sigma^{(n)}h)^{\nu-2}}{\nu}\right). \tag{76}$$

Let us choose K so that

$$\frac{h^2}{x^2}\left\{1 + \sum_{\nu=3}^{4}\frac{(2\lambda\sigma^{(n)}h)^{\nu-2}}{\nu} + \frac{1}{5}\sum_{\nu=5}^{\infty}(2\lambda\sigma^{(n)}h)^{\nu-2}\right\} < \frac{K}{\sigma^{(n)2}} \tag{77}$$

Since, for $0 < \lambda x < \frac{1}{12}$, we have $0 < \lambda\sigma^{(n)}h < \frac{13}{144}$, according to (45) $0 < \frac{\sigma^{(n)}h}{x} < \frac{13}{12}$, and according to (63) formula (77) implies that we can choose

$$K = 1{,}256. \tag{78}$$

The preceding results permit us to formulate the following theorem, which is an improvement of Feller's Theorem 1 stated in [2].

Theorem 4. *Suppose that* X_k, $k = 1, 2, \ldots, n$ *are independent random variables, which satisfy the following conditions: for* $k = 1, 2, \ldots, n$

$$\mathbf{E}(X_k) = 0,$$

$$\mathbf{E}(X_k^2) = \sigma_k^2,\ 0 < \sigma_k^2 < +\infty \tag{79}$$
$$|X_k| < \lambda\sigma,$$

where $\lambda > 0$, $\sigma^2 = \sum_{k\ 1}^{n} \sigma_k^2$.

Suppose further that $0 < \lambda x < \frac{1}{12}$. *Let* $F(z)$ *be the distribution function of the sum* $X = \sum_{k-1}^{n} X_k$; *let*

$$\Phi(z) = \frac{1}{(2\,\pi)^{\frac{1}{2}}} \int_{-\infty}^{z} e^{\ \frac{1}{2}y^2} dy. \tag{80}$$

Then we have

$$1 - F(x\sigma) = e^{-\frac{1}{2}x^2 Q(x)} \{1 - \Phi(x) + \Theta\lambda\, e^{-\frac{1}{2}x^2}\}, \tag{81}$$

where

$$|\Theta| < 7{,}465 \tag{82}$$

$$Q(x) = \sum_{\nu=1}^{\infty} q_\nu x^\nu, \tag{83}$$

the coefficients q_ν *depend only on the* $\nu + 2$ *first moments of* X_k, $k = 1, 2, \ldots, n$, *and they satisfy*

$$|q_1| \leqslant \frac{1}{3}\lambda \tag{84}$$

$$|q_\nu| < \frac{1}{8}(12\lambda)^\nu, \quad \nu = 2, 3, \ldots .$$

Furthermore for every $1 \leqslant i < j \leqslant n$ *we have*

$$|Q^{(j)}(x) - Q^{(i)}(x)| < 1{,}256\, \frac{\sigma^{(j)^2} - \sigma^{(i)^2}}{\sigma^2}, \tag{85}$$

where $Q^{(n)}(x) = Q(x)$, $\sigma^{(n)^2} = \sigma^2$.

Proof. Formulae (81) and (82) are consequences of (71) and (65); (84) comprises formulae (53) and (53′) and (85) is the relation (74).

For $x > 0$ we have

$$1 - \Phi(x) = \frac{1}{(2\,\pi)^{\frac{1}{2}}x} e^{-\frac{1}{2}x^2} \left(1 - \frac{\vartheta}{x^2}\right), \text{ where } 0 < \vartheta < 1. \tag{86}$$

Then (81) can be written as follows

Corollary.

$$1 - F(x\sigma) = \frac{1}{(2\pi)^{\frac{1}{2}} x} e^{-\frac{1}{2}x^2(1+Q(x))} \left\{ 1 - \frac{\vartheta}{x^2} + (2\pi)^{\frac{1}{2}} \Theta \lambda x \right\}, \tag{87}$$

where $|\Theta| < 7{,}465$.

W. Feller in [2] gives the following estimate for (85):

$$|Q^{(j)}(x) - Q^{(i)}(x)| < \frac{1}{2} \frac{\sigma^{(j)^2} - \sigma^{(i)^2}}{\sigma^2}.$$

However, this estimate is not a consequence of the estimate (22) as Feller asserts. For (82) Feller gives the estimate $|\Theta| < 9$ and for (84) the estimate $|q_\nu| < \frac{1}{7}(12\lambda)^\nu$, $\nu = 1, 2, \ldots$; in the statement about q_ν's Feller erroneously states the dependence only on ν first moments (instead of $\nu + 2$).

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