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LINEAR FACTORS IN LATTICE GRAPHS

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1

Let all $\xi_1, \xi_2, \ldots, \xi_n$ be integers > 1, n a positive integer. Similarly as in paper [1] we mean by a lattice graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ a graph(¹) wherein: a) the set of the vertices of the graph is the set V of the points of the Euclidian space E_n, defined in the following way: point x with the co-ordinetes $x_1, x_2, \ldots, x_n$ belongs to V if and only if for all $i = 1, 2, \ldots, n$, x_i is the positive integer $\leqq \xi_i$; b) two vertices from V are in the graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ joined by one single edge if and only if their distance is 1. If the edge h in the lattice graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ joins the vertices $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ and $x_i \neq y_i$, the edge will be said to be parallel to the axis X_i. It is evident that after the removal of all edges from the graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ parallel to the axis X_i, we shall have a graph with ξ_i components. All these components are isomorphic and are called layers of the graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ in the direction of the axis X_i. Let us define the k-th layer in the direction of the axis X_i thus: the vertex $x = (x_1, x_2, \ldots, x_n)$ belongs to the k-th layer of the graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ in the direction of the axis X_i if and only if $x_i = k$. It is evident that the edge joining the vertex $a = (a_1, a_2, \ldots, a_n)$ with the vertex $b = (b_1, b_2, \ldots, b_n)$ belongs to the k-th layer in the direction of the axis X_i if and only if $a_i = b_i = k$.

We shall say that the vertices $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ have a common projection in the direction of the axis X_i, if $x_j = y_j$ for all $j \neq i, j \in \{1, 2, \ldots, n\}$ and we shall say that the edges g, h from $G(\xi_1, \xi_2, \ldots, \xi_n)$ have a common projection in the direction of the axis X_i if neither the edge g nor the edge h are parallel to X_i and if the statement holds that both the one and the other vertex incident at the edge g have a common projection with the vertex incident at the edge h. Similarly: a subgraph or a partial graph or a partial subgraph(²) G_j of

(¹) The difference is that now we admit also $n = 1$, while in paper [1] we assume $n > 1$.

(²) I introduce, analogously with Berge's distinction in the theory of (oriented) graphs, the following distinction between a graph and a subgraph: if I delete from a certain graph G only some of its edges, I shall have the partial graph G'; if I delete from the graph G some of its vertices and besides only the edges incident at those vertices, I shall have the subgraph of the graph (see [2]). If I do not delete anything, I shall have both the subgraph and the partial graph.

the j-th layer in the direction of the axis X_i has a common projection in the direction of the axis X_i with the subgraph or the partial graph or subgraph G_k of the k-th layer in the direction of the same axis if and only if there exists such a simple mapping of the graph G_j on the graph G_k that an element in G_j and its image in G_k have the same projection in the direction of the axis X_i.

Let in the lattice graph $G(\xi_1, \xi_2, \dots \xi_n)$ the edge g join the vertex x with the vertex y and let the edge $\dot{g}$ join the vertex $\dot{x}$ with the vertex $\dot{y}$. The vertices g, $\dot{g}$ are said to be near if (1) $\{x, y\} \cap \{\dot{x}, \dot{y}\} = \emptyset$;(2) in the graph $G(\xi_1, \xi_2 \dots, \xi_n)$ there exists a quadrilateral including the edges g, $\dot{g}$. The remaining two edges of the mentioned quadrilateral will be called the "rungs" of the near edges g, $\dot{g}$. It is evident that for every pair of the near edges there exist in the lattice graph exactly two "rungs" that are near edges, and that the "rungs" of the "rungs" are the two original near edges.

Lemma 1. *Any component of the finite graph with a linear factor has an even number of vertices.*

The proof (which is very simple and can be easily established by the reader himself) is given in paper [3] (see lemma 1).

Theorem 1. *In the lattice graph $G(\xi_1, \xi_2, \dots, \xi_n)$ with an even number of vertices there exist n and only n such linear factors no two of which have a common edge.*

Proof. That in the graph $G(\xi_1, \xi_2, \dots, \xi_n)$ there can exist at most n such linear factors no two of which have a common edge is evident from the fact that the vertex $x = (x_1, x_2, \dots, x_n)$, for which $x_1 = x_2 = \dots = x_n = 1$ is incident at exactly n edges.

Let us prove that there exist n such linear factors of the graph $G(\xi_1, \xi_2, \dots, \xi_n)$. For $n = 1$ the theorem evidently holds, for in the lattice graph $G(\xi_1]$, where $\xi_1 = 2p$, the set of all such edges of the graph that join the vertex $(2i - 1)$ with the vertex $(2i)$; $i = 1, 2, \dots, p$ is the set of the edges of the linear factor of the graph $G(\xi_1)$. Suppose that the theorem holds for all $n \leqq m$; $n \geqq 1$ (where m is an integer > 1) and let us prove that from the aforesaid assumption there follows also the validity of the theorem for $n = m + 1$.

Let $G(\xi_1, \xi_2, \dots, \xi_{m+1})$ be an arbitrary $(m+1)$-dimensional lattice graph with an even number of vertices. As the form of the graph does not depend on the order by which we denote the axes of the co-ordinates, we may suppose without loss of generality that ξ_1 is an even number ($\xi_1 = 2p$). Let us put $\xi_{m+1} = s$ for the sake of simplification and denote by the symbols $G_1, G_2, \dots, G_s$ the first, second, ..., s-th layer of the graph $G(\xi_1, \xi_2, \dots, \xi_{m+1})$ in the direction of the axis X_{m+1}. Each of these layers is isomorphic with the lattice graph $G_0 = G(\xi_1, \xi_2, \dots, \xi_{mr}$ Since $\xi_1 = 2p$, G_1 has an even number of vertices and according to our assumption there exist such linear factors $L_1(1), L_2(1), \dots, L_m(1)$ of the graph G_1, no two of which have a common edge. Let us denote by the symbol $L_i(k)$ such a linear factor of the graph G_k that has a common projection in the direction of the axis X_{m+1} with the linear factor $L_i(1)$ (the existence of such a factor follows from the aforesaid isomorphism of layers). Let us further denote by the symbol $F(k)$ ($k = 1, 2, \ldots, s - 1$) the set of all such edges from $G(\xi_1, \xi_2, \ldots, \xi_{m+1})$ that join the vertex from G_k with the vertex from G_{k+1}.

A. Let s be an even number ($s = 2r$). Let us denote by the symbol L_{m+1}^* a partial graph of the graph $G(\xi_1, \xi_2, \ldots, \xi_{m+1})$ including all edges and only edges of the set

$$\bigcup_{k=1}^{r} F(2k - 1)$$

and let us put $L_j^* = \bigcup_{k=1}^{s} L_j(k)$ for all $j = 1, 2, \ldots, m$. It is evident that for any $k = 1, 2, \ldots, m + 1$ L_k^* is a linear factor of the graph $G(\xi_1, \xi_2, \ldots, \xi_{m+1})$ and no two different linear factors from the aforesaid $m + 1$ linear factors have a common edge. Hence in the case of $s = 2r$ there follows from the validity of the theorem for all $n \leqq m$ its validity for $n = m + 1$.

B. Let s be an odd number; $s = 2r + 1$. Let us denote by the symbol L_m^* the partial graph of the graph $G(\xi_1, \xi_2, \ldots, \xi_{m+1})$, including all edges from $L_m(1)$ as well as all the edges and only edges of the set

$$\bigcup_{k=1}^{r} F(2k).$$

Let us denote by the symbol L_{m+1}^* the partial graph of the same graph containing such and only such edges: all edges from $L_m(s)$ and the edges of the set

$$\bigcup_{k=1}^{r} F(2k - 1).$$

Let it further be true for all $j = 1, 2, \ldots, m - 1$:

$$L_j^* = \bigcup_{k=1}^{s} L_j(k).$$

Each of the partial graphs $L_1^*, L_2^*, \ldots, L_{m+1}^*$ of the graph $G(\xi_1, \xi_2, \ldots, \xi_{m+1})$ is evidently a linear factor of the graph $G(\xi_1, \xi_2, \ldots, \xi_{m+1})$ and no two of them have a common edge. Hence, even in the case of an odd s there follows from the validity of the theorem for $n \leqq m$ its validity for $n = m + 1$. The proof of the theorem is herewith accomplished.

Theorem 2. *Let $G(\xi_1, \xi_2, \ldots, \xi_n)$ be a lattice graph with an even number of vertices and let L be any of its linear factors. For the number $\varrho_k(i)$ of edges from L, joining* *a vertex from the k-th layer with a vertex of the* $(k + 1)$*-th layer in the direction of the axis* X_i, *the following statement evidently holds:*

$$k \prod_{\substack{j=1 \\ j \neq i}}^{n} \xi_j \equiv \varrho_k(i) \pmod 2.$$

Proof. Let us denote by the symbol η_i the number of vertices in one of the layers of the graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ in the direction of the axis X_i. We evidently have:

$$\eta_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} \xi_j.$$

Let us denote by the symbol $M_k(i)$ the set of such edges from L that belong to the k-th layer of the graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ in the direction of the axis X_i and by the symbol $\mu_k(i)$ their number. Let us further denote by the symbol $R_k(i)$ the set of such edges from L that join the vertex from the k-th layer with a vertex of the $(k + 1)$-th layer of the graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ in the direction of the axis X_i; $\varrho_k(i)$ is their number. Let us put further $\varrho_k = 0$ for $k = 0$ and for $k = \xi_i$.

It is evident that: each of the η_i vertices of the k-th layer is incident at exactly one edge of the linear factor L. The number of such these vertices incident at the edge of L belonging to the k-th layer in the direction of the axis X_i is evidently $2\mu_k(i)$. Any of the remaining $\eta_i - 2\mu_k(i)$ vertices is either incident at the edge belonging to $R_{k-1}(i)$ or at the edge belonging to $R_k(i)$. Hence it is true for every $k = 1, 2, \ldots, \xi_i$:

$$\varrho_{k+1}(i) + 2\mu_k(i) + \varrho_k(i) = \eta_i.$$

Whence it follows (consider that $\varrho(i) = 0$):

$$\begin{aligned}
\varrho_1(i) &\equiv \eta_i \pmod 2,\\
\varrho_1(i) + \varrho_2(i) &\equiv \eta_i \pmod 2,\\
\varrho_2(i) + \varrho_3(i) &\equiv \eta_i \pmod 2,\\
\cdots\cdots\cdots\cdots &\\
\varrho_{\xi_i-2}(i) + \varrho_{\xi_i-1}(i) &\equiv \eta_i \pmod 2,\\
+\varrho_{\xi_i-1}(i) &\equiv \eta_i \pmod 2.
\end{aligned}$$

If $\eta_i \equiv 0 \pmod 2$, then all numbers $\varrho_1(i), \varrho_2(i), \ldots, \varrho_{\xi_i-1}(i)$ are even, if $\eta_i \equiv 1 \pmod 2$, then we have:

$$\varrho_1(i) \equiv 1 \pmod 2; \qquad \varrho_2(i) \equiv 0 \pmod 2;$$
$$\varrho_3(i) \equiv 1 \pmod 2; \qquad \varrho_4(i) \equiv 0 \pmod 2;$$

generally: $\varrho_k(i) \equiv k \pmod 2$ for all $k = 1, 2, \ldots, \xi_i - 1$.

Hence the number $\varrho_k(i)$ is odd if and only if both the number k and the number η_i are odd; or: if the product

$$k \prod_{\substack{j=1 \\ j \neq i}}^{n} \xi_j$$

is an odd number.

This proves the theorem.

2

Let L be any linear factor of the lattice graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ and let i be any number from $\{1, 2, \ldots, n\}$. We shall say that two different layers of this graph in the direction of the axis X_i are connected by L if there exists at least one edge from L, joining a vertex of one layer with a vertex of the second layer.

Several well-known problems and their generalizations lead to the concept of the connecting of layers of the lattice graph by its linear factors; these problems will be dealt with subsequently.

Let us consider a chess-board (with its squares of the sides of the length 1) arranged into m columns and n rows. In the case when mn is an even number. we can partition the whole chess-board into $\frac{1}{2}mn$ 1×2 rectangles so that each rectangle includes exactly two squares of the chess-board and each square belongs to exactly one rectangle. The question we shall try to answer is: what conditions must the numbers m, n (giving the dimensions of the chess-board S) fulfil that there exist the above dissection of the chess-board R into 1×2 rectangles in such a way that at each dissection of the chess-board into two oblong chess-boards S_1, S_2, there exists at least one such rectangle of the dissection R that one of its squares belongs to S_1, the other to S_2. Such a dissection will be called the significant dissection of the chess-board into rectangles.

Before solving the above question, we shall express the mentioned problems in the language of the theory of graphs. Let S be an $m \times n$ chess-board. Let us. with respect to the chess-board S, construct the following graph G_S: the vertices of the graph are formed by the squares of the chess-board S and two vertices in G_S are joined by the edge if and only if the respective squares of the chess-board are adjacent, i.e., if they have a common edge. Evidently, G_S is isomorphic with the two-dimensional lattice graph $G(m, n)$.

Lemma 2. *Let S be an $m \times n$ chess-board and let R be such its dissection into 1×2 rectangles that any square in S belongs to exactly one rectangle of the dissection R. Let $G(m, n)$ be a lattice graph whose vertices are the squares oft the chess-board S. and the vertices in $G(m, n)$ are joined by an edge if and only if the respective squares of the chess-board are adjacent. Let us assign to the dissection R the partial graph L_R*

of the graph $G(m, n)$ in the following way: the edge from $G(m, n)$, joining the vertices a, b, belongs to L_R if and only if the squares a, b belong to the same rectangle of the dissection R. Then we have: L_R is the linear factor of the graph $G(m, n)$ and it is true that: the described correspondence of the linear factor of the graph $G(m, n)$ with the dissection of the chess-board into rectangles is a one-to-one map of the set of all dissections of the chess-board S with the required properties on the set of all linear factors of the graph $G(m, n)$.

The proof is evident.

Lemma 3. *Let S be an $m \times n$ chess-board. Let R be its dissection into 1×2 rectangles and let L_R be the corresponding linear factor of the lattice graph $G(m, n)$. The dissection R is the significant dissection of the chess-board S into rectangles if and only if every two adjacent layers of the lattice graph $G(m, n)$ both in the direction of the axis X_1 and the direction of the axis X_2 are connected by L_R.*

Proof. The squares of the $m \times n$ chess-board S are arranged so as to form m columns and n rows. It is possible to cut S into two oblong chess-boards S_1, S_2 either in such a way that all squares of the first p columns ($1 \leqq p < m$) are included in S_1 and the other squares in S_2 (i.e. we cut the chess-board vertically in two), or in such a way that we include all squares of the first q rows ($1 \leqq q < n$) in S_1 and the other squares in S_2 ("horizontal cut"). In the first case there correspond to the chess-boards S_1, S_2 two components of the graph that arises from the graph $G(m, n)$ after we remove all edges joining the vertex of the p-th layer with the vertex of the $(p + 1)$-th layer in the direction of the axis X_1. In the other case we have the components of the graph which arises from $G(m, n)$ after the removal of all edges joining the vertex of the q-th layer with the vertex of the $(q + 1)$-th layer in the direction of the axis X_2.

Let us, once more, denote by the symbol $\varrho_k(i)$ the number of edges from L_R, joining the vertex of the k-th layer with the vertex of the $(k + 1)$-th layer in the direction of the axis X_i ($i = 1, 2$). Evidently, the following is true: R is the singificant dissection of the chess-board into rectangles if and only if $\varrho_j(1) \neq 0$ for all $j = 1, 2, \ldots, m - 1$; $\varrho_k(2) \neq 0$ for all $k = 1, 2, \ldots, n - 1$. The aforesaid conditions are, however, fulfilled if and only if every two adjacent layers of the graph $G(m, n)$ both in the direction of the axis X_1 and the direction of the axis X_2 are connected by L_R. This proves the lemma.

Lemma 4. *Let S be an $m \times n$ chess-board, R its dissection into 1×2 rectangles and let L_R be the linear factor of the lattice graph $G(m, n)$, corresponding to the dissection R. Let us denote by the symbol $\varrho_k(i)$ the number of such edges from L_R that connect the vertex of the k-th layer with the vertex of the $(k + 1)$-th layer of the graph $G(m, n)$ in the direction of the axis X_i ($i = 1, 2$).*

We then have:

$$\sum_{k=1}^{m-1} \varrho_k(1) + \sum_{k=1}^{n-1} \varrho_k(2) = \frac{1}{2} mn, \tag{1}$$

$$\varrho_k(1) \equiv kn \pmod 2 \quad \text{for all} \quad k = 1, 2, \ldots, m-1, \tag{2}$$

$$\varrho_k(2) \equiv km \pmod 2 \quad \text{for all} \quad k = 1, 2, \ldots, n-1. \tag{3}$$

Proof. The validity of the statement (1) of the lemma is evident from the fact that any edge from L_R joins two vertices belonging to different layers either in the direction of the axis X_1, or in the direction of the axis X_2. Statements (2), (3) are a direct consequence of theorem 2.

Let us now deduce this theorem about the significant dissections of the chess-board into rectangles.(3)

Theorem 3. *The significant dissection of the* $m \times n$ *chess-board* ($mn > 2$) *exists if and only if:* (1) *the chess-board has an even number of squares*; (2) $m \geqq 5$; (3) $n \geqq 5$; $m = n = 6$ *does not hold.*

Proof. Let $m = 2p$ and let R be a dissection of the chess-board into 1×2 rectangles. Let L_R be the linear corresponding factor of the lattice graph $G(m, n)$. Let us again denote by the symbol $\varrho_j(1)$, resp. $\varrho_j(2)$ the number of edges from L_R joining the vertex of the j-th layer with the vertex of the $(j+1)$-th layer of the graph $G(m, n)$ in the direction of the axis X_1 or the axis X_2. According to lemma 3, R is a significant dissection of the chess-board into rectangles if and only if:

$$\varrho_j(1) \neq 0 \quad \text{for all} \quad j = 1, 2, \ldots, m-1,$$

and

$$\varrho_j(2) \neq 0 \quad \text{for all} \quad j = 1, 2, \ldots, n-1.$$

I. Suppose that n is on odd number. According to lemma 4 we then have:

$$\varrho_j(1) \equiv j \pmod 2 \quad \text{for all} \quad j = 1, 2, \ldots, 2p-1,$$
$$\varrho_k(2) \equiv 0 \pmod 2 \quad \text{for all} \quad k = 1, 2, \ldots, n-1.$$

If R is a significant dissection of the chessboard S into rectangles, then necessarily:

$$\varrho_i(1) \geqq 1 \quad \text{for all} \quad i = 1, 3, \ldots, 2p-1,$$
$$\varrho_j(1) \geqq 2 \quad \text{for all} \quad j = 2, 4, \ldots, 2p-2,$$
$$\varrho_k(2) \geqq 2 \quad \text{for all} \quad k = 1, 2, \ldots, n-1.$$

(3) S. W. Golomb obtained the same results in paper [4] in a different way.

Hence, as regards the number of edges from L_R (which is np), it is true (see lemma 4):

$$pn = \sum_{i=1}^{p} \varrho_{2i-1}(1) + \sum_{j=1}^{p-1} \varrho_{2j}(1) + \sum_{k=1}^{n-1} \varrho_k(2).$$

Therefore:

$$pn \geqq p + 2(p - 1) + 2(n - 1),$$
$$(p - 2)(n - 3) \geqq 2.$$

According to the supposition n is an odd number.

It is evident that $n \neq 1$, $n \neq 3$. But then necessarily $n \geqq 5$.

The term $(n - 3)$ is always positive; hence it follows that $p - 2 > 0$ and therefore $2p > 5$. The conditions of theorem $m \geqq 5$ and $n \geqq 5$ with an odd n are therefore necessary conditions.

II. Suppose n to be an even number; $n = 2q$.

According to lemma 4 we have:

$$\varrho_j(1) \equiv 0 \pmod 2 \quad \text{for all} \quad j = 1, 2, \ldots, m - 1,$$
$$\varrho_k(2) \equiv 0 \pmod 2 \quad \text{for all} \quad k = 1, 2, \ldots, n - 1.$$

From the condition $\varrho_j(1) \neq 0$; $\varrho_k(2) \neq \mathring{u}$ it follows:

$$\varrho_j(1) \geqq 2 \quad \text{for all} \quad j = 1, 2, \ldots, m - 1,$$
$$\varrho_k(2) \geqq 2 \quad \text{for all} \quad k = 1, 2, \ldots, n - 1,$$

consequently: $2pq \geqq 2(2p + 1) + 2(2q - 1)$, therefore: $(p - 2)(q - 2) \geqq 2$. Wherefrom it evidently follows that $p \geqq 3$, $q \geqq 3$ and so $m \geqq 6$, $n \geqq 6$. It is further evident that we cannot have at the same time $p = 3$, $q = 3$. Each of the conditions (2), (3), (4) of theorem 3 with even m, n is a necessary condition.

III. We can easily see from fig. 1 that with an even m and an odd n the condition $m \geqq 6, n \geqq 5$ is a sufficient condition. Fig. 1 illustrates schematically the construction method of significant chess-board dissections with admissible dimensions into rectangles. To set off the method of construction, the "horizontal" rectangles are hatched.

A similar case, where m and n are even numbers, is illustrated in fig. 2, here the "vertical" rectangles are hatched.

Note. The condition that $mn > 2$ in theorem 3 cannot be omitted, since the chess-board 1×2 can be uniquely partitioned into 1×2 rectangles and this dissection is significant.

3

The considerations discussed in part 2 can be generalized from two-dimensional to n-dimensional chessboards.

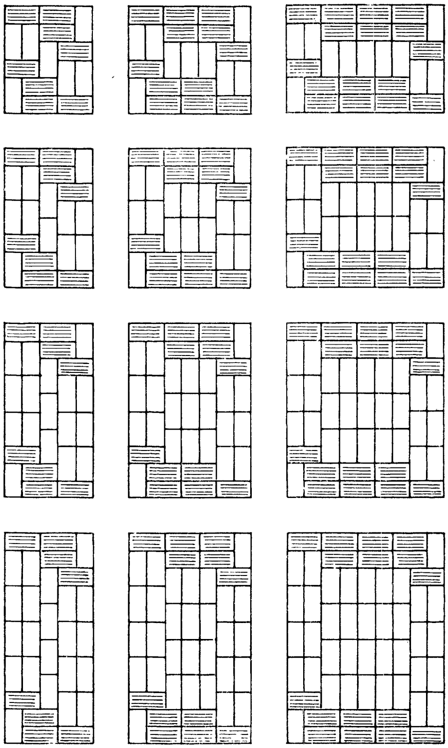

Fig. 1.

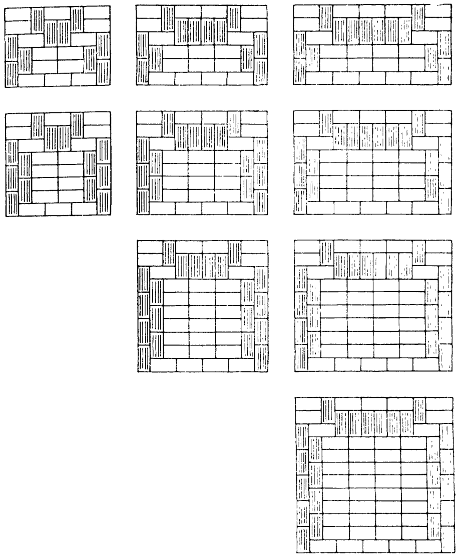

Fig. 2.

Here we shall have an n-dimensional $1 \times 1 \times \ldots \times 1$ cube and the pair of adjacent $1 \times 1 \times \ldots \times 1$ cubes will correspond to the edge of the linear factor. Two such n-dimenional cubes are said to be adjacent if they differ and have a common $(n - 1)$-dimensional cube whose edges are all of the length 1 as well.

Let us deduce the theorem about the existence of the linear factor, by which any two adjacent layers in a lattice graph of more than two dimensions are connected.

Theorem 4. *In the three-dimensional lattice graph $G(\xi_1, \xi_2, \xi_3)$ there exists a linear factor by which any two adjacent layers are connected if and only if the number of its vertices is even and at least two of the numbers ξ_1, ξ_2, ξ_3 are greater than* 2.

Proof. I. Let $\xi_1 = \xi_2 = 2$; $\xi_3 = n \geqq 2$. The number of vertices of the graph $G(2, 2, n)$ is $4n$ and its arbitrary linear factor L has exactly $2n$ edges. Any layer of the graph $G(2, 2, n)$ has evidently an even number of vertices. According to theorem 2 the number of edges from L, connecting any two layers, must be even.

Suppose that any two adjacent layers from $G(2, 2, n)$ are connected by L. Let us denote by the symbol p_i $(i = 1, 2, 3)$ the number of such edges from L that are parallel to the axis X_i. As there exists both in the direction of the axis X_1 and the axis X_2 one and only one pair of adjacent layers, we have $p_1 \geqq 2$; $p_2 \geqq 2$. In the direction of the axis X_3 there exist $n - 1$ pairs of adjacent layers and each of them is connected by at least two edges from L parallel to the axis X_3. Therefore: $p_3 \geqq 2(n - 1)$. Hence $p_1 + p_2 + p_3 \geqq 2 + 2 + 2(n - 1)$. This is a contradiction, since $p_1 + p_2 + p_3 = 2n$. To suppose the existence of the linear factor, by which any two adjacent layers of the graph $G(2, 2, n)$ are connected, leads to a contradiction. The condition that at least two numbers from ξ_1, ξ_2, ξ_3 be greater than 2 is necessary.

II. Let us now prove the following: if at least one number of the numbers ξ_1, ξ_2, ξ_3 is even and at least two of them are greater than 2, then there exists such a linear factor of the graph $G(\xi_1, \xi_2, \xi_3)$, by which any two of its adjacent layers are connected.

A. Let $\xi_1 = 2$, $\xi_2 = 2p + 1$; $\xi_3 = 2q + 1$; $p \geqq 1$; $q \geqq 1$. Fig. 3 illustrates how to find for such a case the linear factor with the required properties. This figure shows the edges of such a linear factor; they belong to the first layer of the graph $G(\xi_1, \xi_2, \xi_3)$ in the direction of the axis X_1 and have a common projection with the edges of the linear factor of the other layer in the direction of the axis X_1. The vertices of this layer, adjacent at such an edge of the linear factor that is parallel

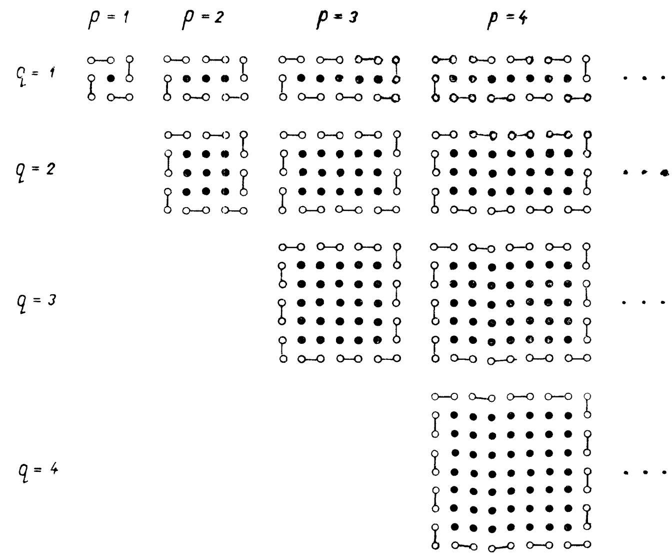

Fig. 3.

with the axis X_1 are represented by full rings, while the other vertices are represented by empty rings. The figure shows the cases where $p = 1, 2, 3, 4$ and $q = 1, 2, 3, 4$ in a way which facilitates the solution of any p, q.

B. Let $\xi_1 = 2$; $\xi_2 = 2p$; $(p > 1)$; $\xi_3 = q$ be any integer greater than 2. Fig. 4 shows, as in A, how to find the linear factor with the required properties.

C. Let all three numbers ξ_1, ξ_2, ξ_3 be greater than 2 and let ξ_1 be an even number, $\xi_1 = 2p$. The linear factor L^*, by which any two adjacent layers of the graph

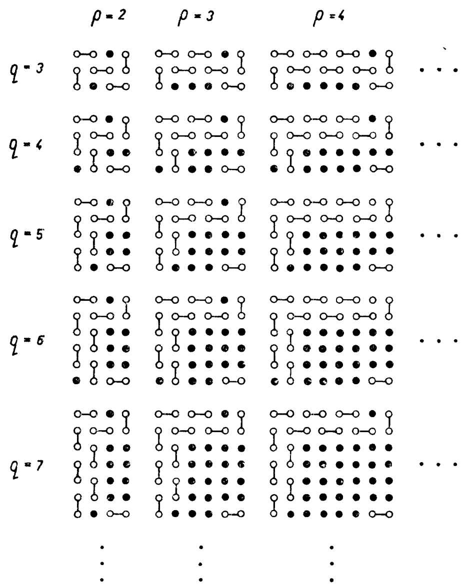

Fig. 4.

$G(\xi_1, \xi_2, \xi_3)$ are connected, will be determined in the following way: We shall find first the linear factor L, by which any two adjacent layers of the graph $G(2, \xi_2, \xi_3)$ ([4]) are connected and let us denote by the symbol H_i the set of such edges from L that belong to the i-th $(i = 1, 2)$ layer of the graph $G(2, \xi_2, \xi_3)$ in the direction of the axis X_1. By the symbol V_0 (or V_1) there will be denoted the set of vertices of this layer, incident (or not incident) at the edge from H_1. The symbol H_{2p} will denote the set of edges from the last layer of the graph $G(\xi_1, \xi_2, \xi_3)$ in the direction of the axis X_1 which has a common projection with the set H_2 in the direction of the axis X_1. Let us form the sets $P_1, P_2, \ldots, P_{2p-1}$ of the edges from $G(\xi_1, \xi_2, \xi_3)$ thus: the edge joining the vertex x from the k-th layer with the vertex belonging to the $(k + 1)$-th layer in the direction of the axis X_1 belongs to P_k if and only if x has a common projection in the direction of the axis X_1 with the vertex belonging to V_j $(j = 0, 1)$, where $j \equiv k \pmod 2$ and P_k is formed only of such vertices. The set

$$H^* = H_1 \cup P_1 \cup P_2 \cup \ldots \cup P_{2-1} \cup H_{2p}$$

([4]) According to A and B such a linear factor evidently exists.

is evidently the set of edges of a certain linear factor L^* of the graph $G(\xi_1, \xi_2, \xi_3)$, by which any two adjacent layers of the graph are connected. This proves the theorem.

Theorem 5. *In every such an n-dimensional lattice graph $G(\xi_1, \xi_2, \ldots, \xi_n)$, wherein the number of vertices is even and n greater than 3, there exists a linear factor, by which any two adjacent layers of the graph are connected.*

Proof. I. Let us describe first the construction of the linear factor, by which any two adjacent layers of a four-dimensional lattice graph are connected.

A. Let $\xi_1 = \xi_2 = \xi_3 = \xi_4 = 2$. Let the linear factor L of the graph $G(2, 2, 2, 2)$ consist of the edges 0000 – 1000, 0111 – 1111, 0001 – 0101, 1010 – 1110, 0100 – 0110, 1001 – 1011, 0010 – 0011, 1100 – 1101, whereby the symbol *abcd – efgh* denotes the edge connecting the vertex (a, b, c, d) with the vertex (e, f, g, h) (fig. 5). Since in the direction of any axis the graph $G(2, 2, 2, 2)$ has exactly two layers and for any $i \in \{1, 2, 3, 4\}$ L contains two edges parallel with the axis X_i, it necessarily follows: any two adjacent layers of the graph $G(2, 2, 2, 2)$ are connected by L.

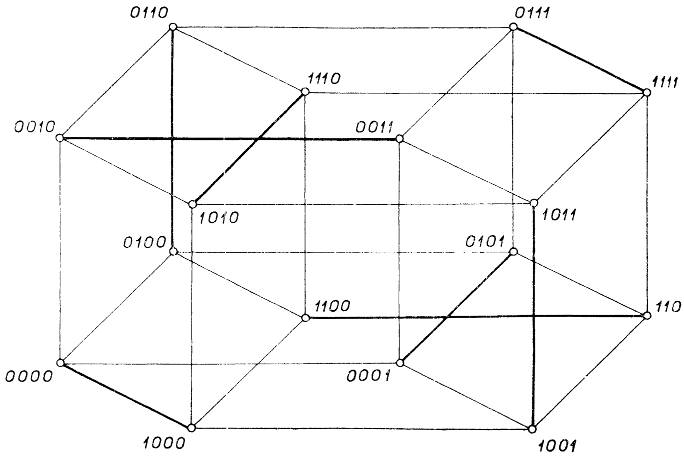

Fig. 5.

B. Let $\xi_1 = \xi_2 = \xi_3 = 2$; $\xi_4 = k \geqq 3$. The way to find the linear factor of the graph $G(2, 2, 2, k)$ by which any two adjacent layers of this graph are connected is given schematically in fig. 6.

C. Let $\xi_1 = \xi_2 = 2$; $\xi_3 = p \geqq 3$; $\xi_4 = q \geqq 3$. Let G_1 (or G_2) be the first (or the second) layer of the graph $G(2, 2, p, q)$ in the direction of the axis X_1. Both these layers are isomorphic with the graph $G(2, p, q)$. According to theorem 4 there exists also a linear factor L_i of the graph G_i by which any two adjacenf layers of the graph G_i are connected ($i = 1, 2$), whereby L_1 and L_2 have a common projection in the direction of the axis X_1. The union $L_1 \cup L_2$ is evidently the linear factor of the graph $G(2, 2, p, q)$. Let h_1 be such an edge from L_1, by which the first layer

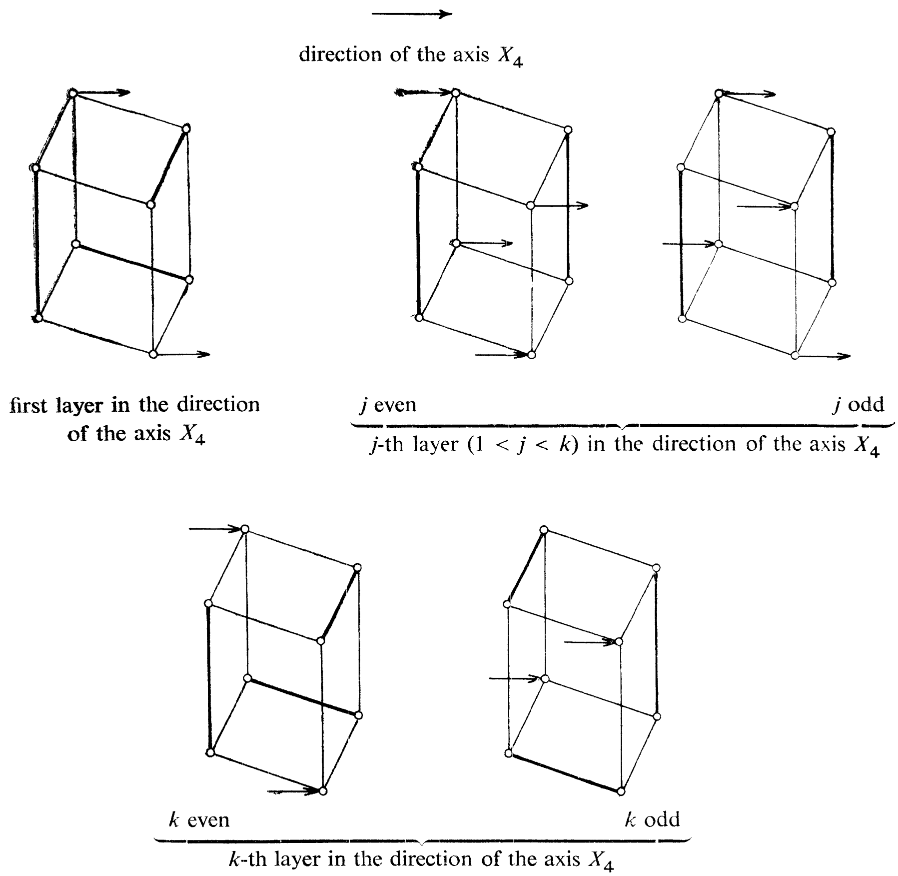

Fig. 6.

of the graph G_1 is connected with the second layer in the direction of the axis X_4. According to theorem 2 there exists an even number of such edges, since the number of vertices of each of the aforesaid layers is $2p$. Hence there exists, besides the edge h_1, at least another edge from L_1, by which the mentioned layers are connected. The linear factor L_2 contains the edge h_2, which has a common projection with the edge h_1 in the direction of the axis X_1. The edges h_1, h_2 belong to adjacent layers: from the above it follows that they are two near edges. If, in the union $L_1 \cup L_2$ we replace the edges h_1, h_2 by their "rungs", we obtain the linear factor of the graph $G(2, 2, p, q)$, by which any two of its adjacent layers are connected.

D. Let $\xi_1 = 2$; $\xi_2 = p \geqq 3$; $\xi_3 = q \geqq 3$; $\xi_4 = r \geqq 3$. Let us denote by the symbols $G_1, G_2, \ldots, G_r$ the first, second, ..., r-th layer of the graph $G(2, p, q, r)$ in the direction of the axis X_4. For all $i = 1, 2, \ldots, r$ the graph G_i is isomorphic with the graph $G(2, p, q)$ and according to theorem 4 there exists a linear factor L_i of the graph G_i, by which any two adjacent layers of the graph G_i are connected. The linear factors $L_1, L_2, \ldots, L_r$ may, with regard to the aforesaid isomorphism, be chosen in such a way that all have a common projection in the direction of the axis X_4. The union $L_0 = L_1 \cup L_2 \cup \ldots \cup L_r$ is evidently the linear factor of the graph $G(2, p, q, r)$, by which every two adjacent layers are connected with the exception of the adjacent layers in the direction of the axis X_4. The number of vertices in any layer of the graph G_i ($i = 1, 2, \ldots, r$) in the direction of the axis X_2 is evidently $2q$ and in the layer of graph G_i in the direction of the axis X_3 this number is $2p$. Hence according to theorem 2 it follows that the number of such edges from L_i, by which any two adjacent layers of the graph G_i are connected in the direction of both the axis X_2 and the axis X_3, is even. Also, this number is greater than zero. It is possible, therefore, to find the edge g_i (or h_i) from L_i for every $i \in \{1, 2, \ldots, r\}$ such that the edges $g_1, g_2, \ldots, g_r$ (or the edges $h_1, h_2, \ldots, h_r$) have a common projection in the direction of the axis X_4 and that by the edge g_i (or h_i), the first and second layer of the graph G_i in the direction of the axis X_2 (or in the direction of the axis X_3) are connected. Hereby g_i (or h_i) is not the only edge by which the above layers are connected. It is evident that the edges g_i, g_{i+1} as well as the edges h_i, h_{i+1} are near. If, therefore, in the union L_0 we replace all the pairs of the near edges g_{2k-1}, g_{2k} ($k = 1, 2, \ldots$; $k \leqq \frac{1}{2}r$) by their "rungs" and if we replace also all pairs of the near edges h_{2k}, h_{2k+1} (where $k = 1, 2, \ldots$; $k < \frac{1}{2}r$) by their "rungs", we shall then evidently have the linear factor of the graph $G(2, p, q, r)$, by which any two of its adjacent layers are connected. (All aforesaid "rungs" are namely parallel with the axis X_4 and the pairs of the adjacent layers in the direction of the axis X_4 are connected by them.)

E. Let $\xi_1 = 2m > 2$; $\xi_2 = p \geqq 3$; $\xi_3 = q \geqq 3$; $\xi_4 = r \geqq 3$. If we removed from the graph $G(2m, p, q, r)$ such edges parallel with X_1 by which the vertex from the $2k$-th layer is joined with the vertex of the $(2k + 1)$-th layer in the direction of the axis X_1 ($k = 1, 2, \ldots, m - 1$), the graph $G(2m, p, q, r)$ would split into m components, each of which would be isomorphic with the graph $G(2, p, q, r)$. Let us denote by the symbol G_i such of these components that includes the $(2i - 1)$-th and the $2i$-th layer of the graph $G(2m, p, q, r)$ in the direction of the axis X_1. According to D there exists in the graph G_i a linear factor L_i, by which any two adjacent layers of the graph G_i are connected. With respect to the aforesaid isomorphism and with regard to D, we can find the linear factors $L_1, L_2, \ldots, L_m$ so that we have: if the edge h from the first layer in the direction of the axis X_1 belongs to L_1. then all edges having a common projection with h in the direction of the axis X_1, belong to $L_0 = L_1 \cup L_2 \cup \ldots \cup L_m$. Let g_k be an arbitrary edge from L_0, belonging to the k-th layer in the direction of the axis X_1 (we shall denote this layer by the symbol F_k) and parallel with the axis X_2 and by which, consequently, the two aforesaid adjacent layers of the graph F_k are connected in the direction of the axis X_2. Besides the edge g_k there exists according to theorem 2 at least another edge from F_k belonging to L_0 by which the two mentioned layers of the graph F_k are connected in the direction of the axis X_2. Then, of course, if we replace in L_0 the near edges g_2, g_3 by their "rungs", further the near edges g_4, g_5 by their "rungs", etc. ..., the near edges g_{2m-2}, g_{2m-1} by their "rungs", we obtain thus the linear factor of the graph $G(2m, p, q, r)$, by which any two of its adjacent edges are connected.

Since at least one of the numbers $\xi_1, \xi_2, \xi_3, \xi_4$ must be even and all are greater than 1; and, since by the change of the order by which we denote the axes of the coordinates nothing is being modified, all cases for $n = 4$ in the cases A, B, C, D, E are included.

II. Let us suppose that the theorem holds for all integers n fulfilling the condition $4 \leqq n \leqq t$ (where t is a certain positive integer) and let us prove that it then holds for $n = t + 1$ as well.

Let $n = t + 1$ and let $G(\xi_1, \xi_2, \ldots, \xi_n)$ be any n-dimensional lattice graph with an even number of vertices. Taking into account the isomorphism, we can assume without loss of generality that the number ξ_1 is an even number $\xi = 2m$.

Each of the $\xi_n = s$ layers of the graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ in the direction of the axis X_n (let us denote them by the symbols $G_1, G_2, \ldots, G_s$; the k-th layer will be denoted by the symbol G_k) is isomorphic with the graph $G(\xi_1, \xi_2, \ldots, \xi_{n-1})$ having an even number of vertices and being at least four-dimensional. Then there exists, according to the assumption, such a linear factor L_k in G_k, by which any two adjacent layers of the graph G_k ($k = 1, 2, \ldots, s$) are connected. The linear factors $L_1, L_2, \ldots, L_s$ can evidently be such that they have a common projection in the direction of the axis X_n.

Since $\xi_1 = 2m$, it necessarily follows that: any layer of the graph G_k in the direction of both the axis X_2 and X_3 has an even number of vertices. According to theorem 2 it then follows that the number of edges from L_k, by which the first and second layer of the graph G_k are connected in the direction of the axis X_2 (as well as in the direction of the axis X_3) is even and > 0. Let us denote by the symbol g_k (or h_k) one of such edges parallel to the axis X_2 (or X_3). A consideration, similar to the one of part I. D will convince us that if in then union $L_1 \cup L_2 \cup \ldots \cup L_s$ we replace the near edges g_1, g_2 by their "rungs", further the edges h_2, h_3 by their "rungs", next the edges g_3, g_4 by their "rungs", etc., we shall finally obtain the linear factor of the graph $G(\xi_1, \xi_2, \ldots, \xi_n)$, by which any two adjacent layers of this graph are connected. Consequently, if the theorem holds for $4 \leqq n \leqq t$, it holds for $n = t + 1$ as well; because it holds for $n = 4$, it holds also for all $n \geqq 4$. This proves the theorem.

4

In this part we shall especially consider near edges in linear factors and a certain transformation of such a linear factor of the lattice graph that includes a pair of near edges.

Lemma 5. *Let L be any such linear factor of the lattice graph $G(\xi_1, \xi_2, \ldots, \xi_n)$, containing two near edges g, h. Let L' be the partial graph of the graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ that arises from L, if we replace the edges g, h by their "rungs", g', h'. Hence: L' is the linear factor of the graph $G(\xi_1, \xi_2, \ldots, \xi_n)$.*

The proof is evident.

If the linear factor L' of the lattice graph arises from the linear factor L of this graph in such a way that we replace certain two near edges g, h by their "rungs" g', h', we say that L' arises by a κ-transformation of L on the edges g, h.

Lemma 6. *Let the linear factor L' of a certain lattice graph arise by the κ-transformation of the linear factor L of the graph G on its edges g, h, and let g', h' be the "rungs" of the edges g, h. In that case the following is true: The linear factor L is obtained by the κ-transformation of L' on the edges g', h'.*

The proof is evident.

Lemma 7. *Any linear factor of a two-dimensional lattice graph contains at least one pair of near edges.*

Proof. If in the graph $G(m, n)$ there exists a linear factor L not including any pair of near edges, then there belongs to L from each of the $(m - 1)(n - 1)$ quadrilaterals of the graph $G(m, n)$ at most one side. Let s quadrilaterals contain a boundary edge (i.e., an edge from the first or last layer in the direction of some axis) belonging to L. Let us count the number of edges from L considering the individual quadrilaterals. Since all edges from L with the exception of the S boundary edges will be included in exactly two quadrilaterals, we count these edges with the coefficient $\frac{1}{2}$. Since the total number of edges from L is $\frac{1}{2}mn$, we obtain:

$$s + \tfrac{1}{2}[(m - 1)(n - 1) - s] \geqq \tfrac{1}{2}mn,$$

i. e. $s + (m - 1)(n - 1) \geqq mn$ and since evidently $(m - 1) + (n - 1) \geqq s$ then a fortiori $(m - 1) + (n - 1) + (m - 1)(n - 1) \geqq mn$ whence after adjustment we have $-1 \geqq 0$, which is a contradiction. This proves the lemma.

Theorem 6. *Let $G(\xi_1, \xi_2)$ be any two-dimensional lattice graph and let L, L' be any two its linear factors, then L' is obtained from L by the finite number of κ-transformations.*

Proof. Since the product ξ_1, ξ_2 is an even number, we may suppose without loss of generality ξ_1 to be an even number. Let us put, for the sake of simplification of notation $\xi_1 = 2m$; $\xi_2 = n$. From lemma 6 it follows that: To prove the validity of the theorem, it is sufficient to prove that for any linear factor L of the graph $G(2m, n)$ there exists a finite sequence $L_1, L_2, \ldots, L_s$ of linear factors of the graph $G(2m, n)$ so that $L_1 = L$ and that L_s is a linear factor, containing such and only such edges that join the vertex $(2j - 1, k)$ with the vertex $(2j, k)$, where $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, n$; and where the following holds: L_{i+1} is obtained by κ-transformations of the linear factor L_i on certain of its near edges.

We shall prove the statement by induction with respect to n (with m fixed). Before carrying on our discussion, we shall prove the validity of the statement for all graphs $G(2m, 2)$.

I. Suppose that any linear factor of the graph $G(2m, n)$ (where m is a positive integer; $n > 1$; $n \leqq p$ and where p is an integer >1) can by a repeated κ-transformation be converted into a linear factor, whose edges are all parallel with the axis X_1. Let L be any linear factor of the graph $G(2m, p + 1)$.

Denote by the symbol $W(L)$ the set of the vertices of the $(p + 1)$-th layer of the graph $G(2m, p + 1)$ in the direction of the axis X_2 that are incident at an edge from L parallel with the axis X_2. If $W(L)$ were a void set, it would not be necessary to prove anything, since in that case all edges from L, incident at the vertex of the $(p + 1)$-th layer of the graph $G(2m, p + 1)$ in the direction of the axis X_2 are parallel with the axis X_1, and the edges from L, incident at other vertices, form the linear factor L_0 of the graph $G(2m, p)$, which, according to the supposition, can be converted by repeated κ-transformations into the linear factor of the graph $G(2m, p)$ containing only edges parallel with X_1.

Let $W(L)$ be a non-empty set, $W(L) = \{(a_1, p + 1), (a_2, p + 1), \ldots, (a_q, p + 1)\}$, where $q > 0$ and where $a_1 < a_2 < \ldots < a_q$.

A. I maintain that: by κ-transformations, L can be converted into such a linear factor L^*, for which it is true: $W(L^*)$ does not contain any of the vertices $(1, p + 1)$, $(2, p + 1), \ldots, (a_1, p + 1)$. Let us prove the validity of this assertion. Let us form the sequence $V = \{v_1, v_2, \ldots, v_t\}$ of the vertices from $G(2m, p + 1)$ such that: $v_k = (a_1 + k - 1, p - k + 2)$ for all $k = 1, 2, \ldots, t$ where t is chosen so as to be the greatest integer fulfilling the following two conditions:

$$t < 2m + 2 - a_1; \qquad t < p + 2.$$

The following statement holds: in the sequence V there exists at least one such vertex, from which we cannot proceed downward along the edge from L. Let us suppose, conversely, that from each vertex from V we can proceed downward along the edge from L (i.e., to the vertex, belonging to the lower layer of the graph in the direc tion of the axis X_2). Then evidently the vertex from the first layer in the direction of the axis X_2 does not belong to V and the vertex v_t belongs to the last

(2m-th) layer of the graph $G(2m, p + 1)$ in the direction of the axis X_1 ($t = 2m + 1 - a_1$).

Let f_i be an edge from L incident at the vertex v_i. It follows from the above supposition that the vertex $w_1 = (a_1 + 1, p + 1)$ is joined by an edge from L with the vertex $(a_1 + 2, p + 1)$, since it cannot be joined by such an edge either with the vertex v_1 (incident at the edge $f_1 \in L$) or the vertex v_2, incident at the edge $f_2 \in L$. But in such a case the vertex $w_2 = (a_1 + 2, p)$ is joined by an edge from L with the vertex $(a_1 + 3, p)$ (there is no other possibility – fig. 7).

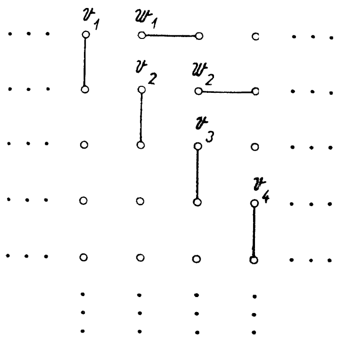

Fig. 7.

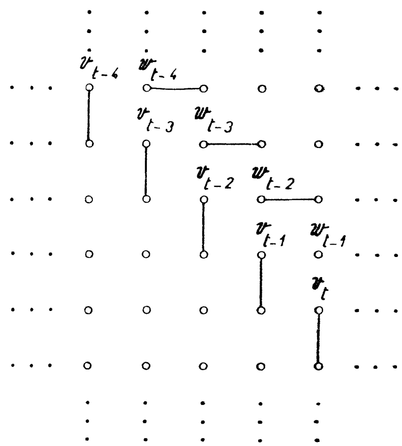

Fig. 8.

Carrying on the above discussion, we find that for any $k = 1, 2, \ldots, t - 2$ it is true: the vertex $w_k = (a_1 + k, p - k + 2)$ is joined by an edge from L with the vertex $(a_1 + k + 1, p - k + 2)$. But then the vertex $w_{t-1} = (2m, p - t + 3)$ cannot be incident at any edge from L (fig. 8).

The supposition that from each vertex of the sequence V we reach along the edge from L the lower layer of the graph $G(2m, p + 1)$ in the direction of the axis X_2, leads to a contradiction.

Let v_c be the first such vertex of the sequence V from which we cannot reach along the edge from L the lower layer of the graph in the direction of the axis X_2. It is evident that for each linear factor F of the graph $G(2m, p + 1)$, for which $W(F) \neq \emptyset$, the number c is uniquely determined; let us denote this number by the symbol $\gamma(F)$, if $W(F) \neq \emptyset$ and let us put $\gamma(F) = 0$, if $W(F) = \emptyset$.

In this case it is evidently true that $\gamma(L) = c > 1$ and that: from the vertex v_c we reach along the edge from L either the higher layers in the direction of the axis X_1 (the first type of linear factor), or the higher layer in the direction of the axis X_2 (the second type). A consideration similar to the one above will convince us easily that the position of the edges from L, incident at the vertices $v_1, v_2, \ldots, v_c$ and the vertices $w_k = (a_1 + k, p - k + 2)$, in case of the first (or the second) type of the linear factor of the graph $G(2m, p + 1)$ is that given in fig. 9a (or fig. 9b).

Whether we have the first or the second type of the linear factor, it is always true that: for an edge from L (let us denote it by h), incident at the vertex v_c there exists such an edge $g \in L$ that is near to the edge h. Besides, we have for the second type $g = f_c$. Let L° be a linear factor of the graph $G(2m, p + 1)$, which arises by an

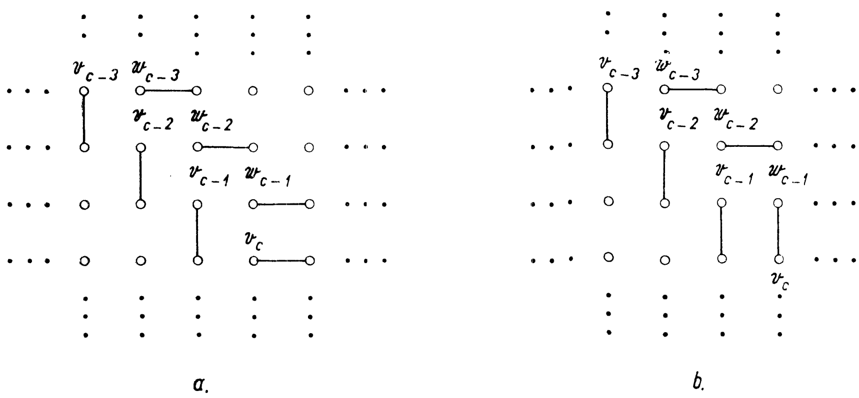

Fig. 9.

κ-transformation of the linear factor L on the edges g, h. The following is true: if L is the first type of the linear factor, then L° is the second type and $\gamma(L) = \gamma(L^\circ)$; if L is the second type of the linear factor, then L° is the first type and we have: $\gamma(L^\circ) = \gamma(L) - 1$ (fig. 9 a, b). Whence it follows: by κ-transformations on near edges from which one is always incident at the vertex $v_{\gamma(L_x)}$ we can change the type of the linear factor and successively reduce the value of the function $\gamma(L)$ so that we finally have a linear factor L^*, for which is it true that: an edge from L^*, incident at the vertex v_1, is incident at the vertex $(a_1 + 1, p + 1)$ (i.e., horizontal – fig. 10). But then the set $W(L^*)$ evidently does not contain any of the vertices $(k, p + 1)$, where $k = 1, 2, \ldots, a_1$, which proves the validity of statement A.

B. I maintain: by κ-transformations the linear factor L can be converted into such a linear factor L^{**}, for which it is true: $W(L^{**})$ is a void set. Let us prove it! Let L_x be any linear factor of the graph $G(2m, p + 1)$. Let us use, for the sake of simplification, the symbol b_i to denote the vertex $(i, p + 1)$ and let us denote by $\beta(L_x)$ such a smallest index j from $\{1, 2, \ldots, 2m\}$ for which it is true: b_j belongs to $W(L)$. According to point A any linear factor $L = L(0)$, for which $W(L(0)) \neq \emptyset$, can by κ-transformations be converted into the linear factor $L(1)$ in such a way that we either have $W(L(1)) = \emptyset$ or $\beta[L(0)] < \beta[L(1)]$. Generally: if for the linear factor $L(k)$ it is true that $W(L(k)) \neq \emptyset$, we can convert this linear factor by κ-transformations into the linear factor $L(k + 1)$ so that we either have $W(L(k + 1)) = \emptyset$, or that it is true: $\beta[L(k)] < \beta[L(k + 1)]$. Hence there always exists a finite sequence of the linear factors $L(0)$, $L(1)$, $\ldots$, $L(r)$ so that $L(i + 1)$ arises by κ-transformations from $L(i)$

and it is true: $\beta[L(0)] < \beta[L(1)] < \ldots < \beta[L(r-1)]$; $W(L(r)) = \emptyset$. But then $L^{**} = L_r$ is the required linear factor. This proves the validity of the assertion B.

C. From the validity of the assertion B there follows – as mentioned at the beginning – that: if our theorem is valid for all lattice graphs $G(2m, n)$ with a given

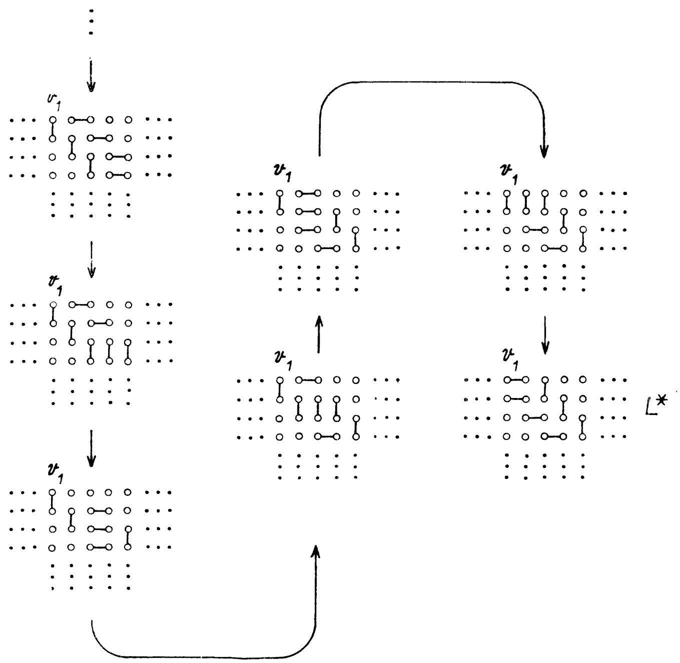

Fig. 10.

$m > 0$ and any $n > 1$; $n \leqq p$, then the theorem holds for the graph $G(2m, p+1)$ as well. To complete the proof of the theorem, it is sufficient to prove merely that the theorem holds for every graph $G(2m, 2)$, where m is any positive integer.

II. Let L be an arbitrary linear factor of the graph $G(2m, 2)$, where m is any integer >0. According to theorem 2 the number of edges from L that are joining the vertex from the k-th layer ($k = 1, 2, \ldots, 2m-1$) of the graph $G(2m, 2)$ in the direction of the axis X_1 with the vertex of the $(k+1)$-th layer of this graph in the direction of the axis X_1 is an even number. Hence the vertices of the considered two layers are either not joined by any edge from, or are joined by exactly two edges from L. Then necessarily: for each such edge from L that is parallel with the axis X_1 and belongs to the first layer in the direction of the axis X_2, there exists a near edge in the second layer of the graph $G(2m, 2)$ in the direction of the axis X_2. Whence it immediately follows that by κ-transformation on each of such pairs of near edges of the linear factor L that are parallel with the axis X_1, we can obtain the linear factor E, all edges of which are parallel with the axis X_2. Let us denote by the symbol e_k the edge from E that connects the vertices $(k, 1)$, $(k, 2)$; $k = 1, 2, \ldots, 2m$. If we perform the κ-transformation first on the edges e_1, e_2, then on the edges e_3, e_4, and so on, ..., on the edges e_{2m-1}, e_{2m}, we obtain the linear factor L_s, all edges of which are parallel with the axis X_1. The theorem then holds for all graphs $G(2m, 2)$, where m is a positive integer.

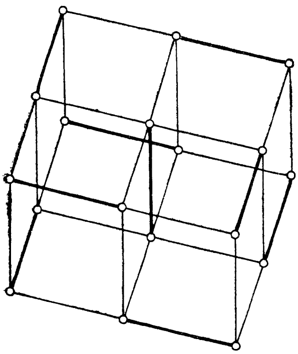

Fig. 11.

Then, according to part I the theorem holds for all $G(2m, n)$, where m, n are integers, $m > 0$, $n > 1$. This was to be proved.

I must make clear that the theorem analogous to theorem 6 for n-dimensional lattice graphs does not hold any more for $n = 3$. Thus for instance in the graph $G(3, 3, 2)$ there exists a linear factor L that does not contain any pair of near edges (fig. 11; the edges from L are set off by bold lines).

The graph of the four-dimensional cube $G(2, 2, 2, 2)$ may even be decomposed into four linear factors L_1, L_2, L_3, L_4 so that there does not exist in any of these four linear factors a pair of near edges. Let us demonstrate at least one example of such a decomposition (fig. 12):

Theorem 7. *Let p be any positive integer. The graph of the 4p-dimensional cube can be decomposed into 4p linear factors so that not one of the linear factors of this decomposition contains a pair of near edges. If there exists the decomposition of the graph of an n-dimensional cube ($n > 1$) into n-linear factors, not one of which contains the pair of near edges, then there exists also the decomposition of the graph of the ($n + 4p$)-dimensional cube into ($n + 4p$) linear factors, not one of which contains two near edges.*(5)

Edges from the linear factor			
L_1	L_2	L_3	L_4
join these pairs of vertices			
0000,1000	0000,0100	0000,0010	0000,0001
0100,0110	1000,1001	1000,1100	1000,1010
0010,0011	0010,1010	0100,0101	0100,1100
0001,0101	0001,0011	0001,1001	0010,0110
1110,1010	1110,1100	1110,0110	1101,1001
1101,1100	1101,0101	1011,1010	1011,0011
1011,1001	0111,0110	0111,0011	0111,0101
1111,0111	1111,1011	1111,1101	1111,1110

(5) When speaking of the graph of an n-dimensional cube we mean the graph $G(2, 2, \ldots, 2)$ (n twos).

Proof. Let $C(n)$ be the graph of an n-dimensional cube and let there exist such its decomposition $R(n) = \{L_1(n), L_2(n), \ldots, L_n(n)\}$ into linear factors that no linear factor of this decomposition contains a pair of near edges. Let $R(4) = \{L_1(4),$

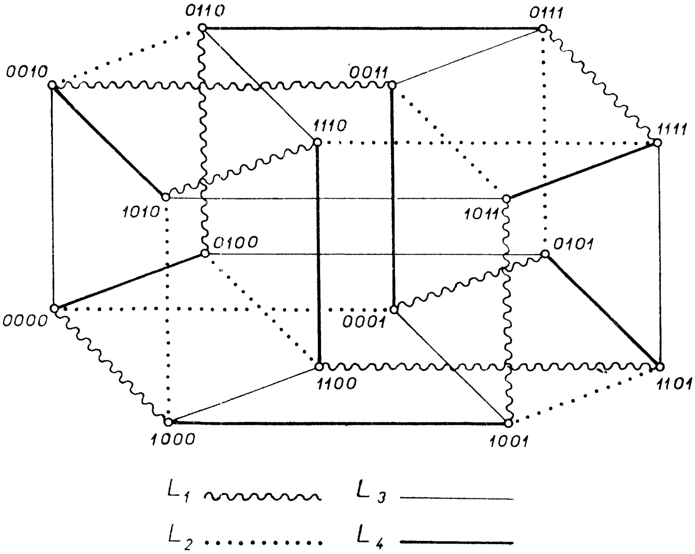

Fig. 12.

$L_2(4), L_3(4), L_4(4)\}$ be the decomposition of the graph of the four-dimensional cube $C(4) = G(2, 2, 2, 2)$ described in table 1 and illustrated in fig. 12. Let V be the set of all vertices of the $n + 4$-dimensional cube $C(n + 4)$ and let $\overline{V} = \{V_{00}, V_{01}, V_{10}, V_{11}\}$ be the decomposition of the set V into classes, defined in the following way: the vertex $(x_1, x_2, \ldots, x_{n+4}) = x$ belongs to the class $V_{i,j} \in \overline{V}$ $(i = 0, 1;\ j = 0, 1)$ if and only if it is true:

$$\sum_{k=1}^{n} x_k \equiv i \pmod 2; \qquad \sum_{k=1}^{4} x_{k+n} \equiv j \pmod 2.$$

Let H be the set of all edges from $C(n + 4)$ and let $\overline{H} = \{H_{0*}, H_{1*}, H_{*0}, H_{*1}\}$ be its decomposition thus defined: the edge h from H, joining the vertex x with the vertex y belongs (fig. 13) to the class:

$$H_{0*} \text{ if and only if } \{x, y\} \cap V_{00} \neq \emptyset;\ \{x, y\} \cap V_{01} \neq \emptyset,$$
$$H_{1*} \text{ if and only if } \{x, y\} \cap V_{10} \neq \emptyset;\ \{x, y\} \cap V_{11} \neq \emptyset,$$
$$H_{*0} \text{ if and only if } \{x, y\} \cap V_{00} \neq \emptyset;\ \{x, y\} \cap V_{10} \neq \emptyset,$$
$$H_{*1} \text{ if and only if } \{x, y\} \cap V_{01} \neq \emptyset;\ \{x, y\} \cap V_{11} \neq \emptyset.$$

It is evident that $\bar{H}$ is a partition, i.e., any edge from $C(n + 4)$ belongs to exactly one set from $\bar{H}$ and each set from the sets H_{0*}, H_{1*}, H_{*0}, H_{*1} is a non-empty set.

Let us denote the already defined linear factors of the graph $C(n)$ (or $C(4)$) in the following way: $L'_i(n) = L_{i-1}(n)$, where $L_0(n) = L_n(n)$ and $L'_i(4) = L_{i-1}(4)$ and where similarly $L_0(4) = L_4(4)$.

Let $R^* = \{L_1^*, L_2^*, \dots, L_{n+4}^*\}$ be the decomposition of the graph $C(n + 4)$ into linear factors, defined in the following way:

1. the edge from H_{0*} joining the vertex $x = (x_1, x_2, \dots, x_{n+4}) \in V_{00}$ with the vertex $y = (y_1, y_2, \dots, y_{n+4}) \in V_{01}$ belongs to L_{n+i}^* if and only if the edge from $C(4)$ joining the vertex $\dot{x} = (x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4})$ with the vertex $\dot{y} = (y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4})$ belongs to $L_i(4)$;

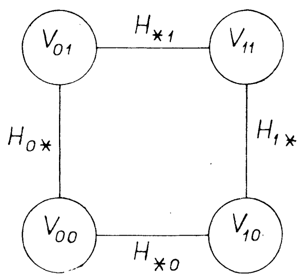

Fig. 13.

2. the edge from H_{1*} connecting the vertex x with the vertex y belongs to L_{n+1}^* if and only if the edge from $C(4)$ joining the vertex $\dot{x}$ with the vertex $\dot{y}$ belongs to $L'_i(4)$;

3. the edge from H_{*0} joining the vertex x with the vertex y belongs to L_j^* if and only if the edge from $C(n)$ joining the vertex $\bar{x} = (x_1, x_2, \dots, x_n)$ with the vertex $\bar{y} = (y_1, y_2, \dots, y_n)$ belongs to $L_j(n)$.

4. the edge from H_{*1} joining the vertex x with the vertex y belongs to L_j^* if and only if the edge from $C(n)$ joining the vertex $\bar{x}$ with the vertex $\bar{y}$ belongs to $L'_j(n)$.

From the above description it is evident that R^* is a decomposition into linear factors and that no linear factor from R^* contains a pair of near edges.

Hence, if by the required way the graph $C(n)$ can be decomposed into linear factors, the graph $C(n + 4)$ can be decomposed in this way as well. Whence there instantly follows the validity of both statements of the theorem.

I wish to make clear that the requirement expressed in theorem 7 with respect to the linear factors of the decomposition, i.e. the requirement that not one of them contain a pair of near edges, can be made more conspicuous and we can postulate that it be true: each of the four edges of any quadrilateral of the graph belongs to a different linear factor of the decomposition (we have omitted in our considerations the case $n = 1$, it being a trivial case). Hence it is clear that the required decomposition cannot exist for $n = 2$, $n = 3$. We can prove that such a decomposition does not exist for $n = 5$ either (I did not succeed in symplifying the rather complicated proof I established). The problem of the existence of these decompositions for such $n > 5$ that are not divisible by 4 remains to be solved.

5

In this part we shall deal with certain properties of the sets of edges in infinite n-dimensional lattice graphs and we shall deduce, with respect to them, a theorem, whose deductions have an importance also when we consider linear factors in finite lattice graphs. When speaking of an n-dimensional infinite lattice graph we think – as in paper [5] – of a graph, whose vetices are all the points from E_n, each coordinate of which is an integer; any two vertices in the graph are joined by an edge if and only if their distance is 1. We use the symbol $G[n]$ to denote it.

Let us define the set of $Y[n]$ vertices from $G[n]$ thus: the vertex $x = (x_1, x_2, ..., x_n)$ from $G[n]$ belongs to $Y[n]$ if and only if

$$\sum_{i=1}^{n} x_i \equiv 0 \pmod 2.$$

Lemma 8. *Any edge from $G[n]$ joins a vertex from $Y[n]$ with a vertex not belonging to $Y[n]$.*

The proof is evident (the statement in the lemma follows directly from the definition of the graph $G[n]$).

The partial graph F of the graph $G[n]$ will be said to be a Λ-graph in $G[n]$, if it is true: any vertex from $G[n]$ is incident at most with one edge from F. We evidently have: any linear factor (hence also the partial graph of the linear factor) of the graph $G[n]$ is its Λ-graph.

Let g, h be any two edges from $G[n]$ and let the edge g connect the vertices v, w; the edge h the vertices x, y. The edges g, h will be said to be adjacent if $g \neq h$ and if it is true that $\{v, w\} \cap \{x, y\} = \emptyset$ and almost adjacent if $g \neq h$; g, h are not adjacent edges and if in the graph $G[n]$ there exists such an edge f that both (f, g) and (f, h) are pairs of adjacent edges.

Lemma 9. *Let g, h be any two almost adjacent edges from $G[n]$ and let $v = (v_1, v_2, ..., v_n)$, or $w = (w_1, w_2, ..., w_n)$ be the vertex from $Y[n]$ at which the edge g, or the edge h is incident. In such a case we have:*

$$\sum_{i=1}^{n} | v_i - w_i | = 2.$$

Proof. Let $\dot{v}$ (or $\dot{w}$) be the vertex from $G[n]$ not belonging to $Y[n]$, at which the edge g (or edge h) is incident – see lemma 8 – and let f be the edge joining the vertex from $\{v, \dot{v}\}$ with the vertex from $\{w, \dot{w}\}$. We may suppose without loss of generality that the edge f joins the vertex v with the vertex $\dot{w}$. Taking the definition of the graph $G[n]$ as a starting point, we have, after a simple consideration:

$$\sum_{i=1}^{n} | v_i - \dot{w}_i | = 1; \qquad \sum_{i=1}^{n} | \dot{w}_i - w_i | = 1.$$

Whence it follows that there exist such numbers $r, s \in \{1, 2, ..., n\}$ that:

$$| v_r - \dot{w}_r | = 1; \quad v_i = \dot{w}_i \text{ for all } i \neq r; \ i \in \{1, 2, ..., n\},$$
$$| \dot{w}_s - w_s | = 1; \quad \dot{w}_i = w_i \text{ for all } i \neq s; \ i \in \{1, 2, ..., n\}.$$

According to our assumption v, w are two different vertices. Whence it follows that: if $r = s$, then necessarily $| v_r - w_r | = 2$; $v_i = w_i$ for all $i \neq r$; $i \in \{1, 2, ..., n\}$; and if $r \neq s$, then it is true: $| v_r - w_r | = 1$; $| v_s - w_s | = 1$; $v_i = w_i$ for all $i \in \{1, 2, ..., n\}$ not belonging to $\{r, s\}$. The validity of the lemma from the the aforesaid is evident.

Let F be any A-graph in $G[n]$. We say that F can be coloured by p colours (p is a positive integer), if to each edge from F a number from $\{1, 2, ..., p\}$ cen be assigned (this number will be called the colour of the edge) so that any two almost adjacent edges will have a different colour.(6)

Theorem 8. *Every A-graph in the graph $G[n]$ can be coloured by p colours, where $p \leqq 2n$.*

Proof. Let F be any A-graph in the graph $G[n]$. Let $\overline{M} = \{M_1, M_2, ..., M_{2n}\}$ be the partition of the set $Y[n]$ into classes thus defined: the vertex $x = (x_1, x_2, ..., x_n)$ from $Y[n]$ belongs to the class M_i ($i = 1, 2, ..., n$) of the partition $\overline{M}$ if and only if:

$$\mu(x) = \frac{1}{2}[x_1 + 3x_2 + ... + (2n - 1)\, x_n] \equiv i (\bmod 2n).$$

A. I assert: if it is true for the vertices x, y from $Y[n]$ that:

$$\sum_{i=1}^{n} | x_i - y_i | = 2, \tag{*}$$

then the vertices x, y belong to different classes of the partition $\overline{M}$. Let us prove it! Let (*) hold for the vertices $x, y \in Y[n]$. It is evident that only the following two cases are possible:

Case I. There exists such a number $q \in \{1, 2, ..., n\}$ that $| x_q - y_q | = 2$ and $x_i = y_i$ holds for all $i \neq q$; $i \in \{1, 2, ..., n\}$.

Case II. There exist numbers $r < s$ belonging to $\{1, 2, ..., n\}$ so that $| x_r - y_r | = 1$; $| x_s - y_s | = 1$; $x_i = y_i$ holds for all $i \neq r$; $i \neq s$.

In the first case we have:

$$\mu(y) = \frac{1}{2}[y_1 + 3y_2 + ... + (2n - 1)\, y_n] =$$

$$= \frac{1}{2}[\pm(4q - 2) + x_1 + 3x_2 + ... + (2n - 1)\, x_n]$$

or

$$\mu(y) = \mu(x) \pm (2q - 1)$$

and since the number $2q - 1$ cannot be the integer multiple of the number $2n$ it follows: the vertices x, y belong to different classes of the partition $\overline{M}$.

(6) It follows directly from the definition of the A-graph that F cannot contain two adjacent edges.

In the second case we have:

$$\mu(y) = \mu(x) + \frac{1}{2}[\pm(2r - 1) \pm (2s - 1)] =$$

$$= \mu(x) + \left[\pm\left(r - \frac{1}{2}\right) \pm \left(s - \frac{1}{2}\right)\right] = \mu(x) + v.$$

The term in brackets (denoted by the symbol v) cannot be equal to zero since, according to the assumption, $r < s$. Since $r < s \leqq n$, it is necessarily true that $v < 2n$. Hence v is not the integer multiple of the number $2n$ and so the vertices x, y must belong to different classes of the partition $\overline{M}$. This proves the validity of our assertion.

Let h be any edge from F and let $y(h)$ be such of the vertices incident at the edge h that belongs to $Y[n]$ (see lemma 8). The edges from F shall be coloured in the following way: the edge h has the colour i ($i = 1, 2, \ldots, 2n$) if and only if the vertex $y(h)$ belongs to the class M_i of the partition $\overline{M}$.

B. I assert: With the above colouring of edges from F any two almost adjacent edges from F are coloured differently. Let us prove the validity of the assertion! Let g, h be any two almost adjacent adges from F. Let $v = y(g)$, or $w = y(h)$ be the vertex from $Y[n]$, at which the edge g, or the edge h, is incident. According to lemma 9 it is true:

$$\sum_{i=1}^{n} | v_i - w_i | = 2$$

whence it follows according to A that: the vertices $v = y(g)$; $w = y(h)$ belong to different classes of the decomposition $\overline{M}$. Hence it bocomes evident that with the mentioned colouring of the edges from F, the edges g, h are differently coloured. This proves the theorem.

A direct consequence of theorem 8 is the following theorem:

Theorem 9. *Each linear factor of the lattice graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ can be coloured by p colours, where $p \leqq 2n$.*

Proof. Any linear factor of the lattice graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ is evidently a Λ-graph in the graph $G[n]$. According to theorem 8, it can be coloured by $p \leqq 2n$ colours.

Let us turn once more to the dissections of the two-dimensional chess-board into 1×2 quadrilaterals, or to the dissections of the n-dimensional chess-board into pairs of adjacent n-dimensional cubes with edges of the length 1. Two n-dimensional cubes with edges of the length 1, i. e., two cubes of the n-dimensional chess-board, are adjacent if they are different and if their intersection in an $(n - 1)$-dimensional cube with edges of the length 1.

Let S be an n-dimensional $\xi_1 \times \xi_2 \times \ldots \times \xi_n$ chess-board and let $\overline{D} = \{D_1, D_2, \ldots, D_m\}$ be such a dissection of the chess-board S into pairs of adjacent

n-dimensional cubes with edges of the length 1 that we have: each cube of the chess-board S belongs to exactly one pair from $\bar{D}$. Let L be the linear factor of the lattice graph $G(\xi_1, \xi_2, \ldots, \xi_n)$ corresponding to the dissection $\bar{D}$. Let further $D_i \neq D_j$ be any two pairs from $\bar{D}$ and let h_i, h_j be their corresponding edges from L.

Lemma 10. *The cube of the chess-board S belonging to D_i and adjacent at a certain cube from D_j exists if and only if the edges h_i, h_j are almost adjacent.*

The proof follows directly from the definition of the almost adjacent edges and from the correspondence between an n-dimensional lattice graph and an n-dimensional chess-board.

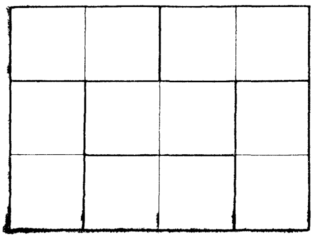

Fig. 14.

From lemma 10 it immediately follows: The problem to colour the cubes of an n-dimensional chess-board with the given dissection $\bar{D}$ with p colours (so that both cubes belonging to the pair have the same colour and that each two pairs from $\bar{D}$, the intersection of which is at least a $(n - 1)$-dimensional cube with edges of the length 1, have a different colour) is therefore equivalent to the problem how to colour the edges of the linear factor L with p colours (so that each two almost adjacent edges have a different colour). The following theorem then holds, which may be called the small four-colour theorem.

Theorem 10. *Let S be any two-dimensional chess-board and let $\bar{D}$ be any its dissection into 1×2 rectangles. The rectangles from $\bar{D}$ can always be coloured with the help of four colours so that any two rectangles, whose common boundary is formed by a line-segment of the length of at least 1, have a different colour.*

Proof. The theorem follows directly from theorem 9 for the special case $n = 2$.

If the chess-board S from theorem 10 is a $2m \times n$ chess-board where $m \geqq 2$, then there exists such its dissection into 1×2 rectangles that we need for their colouring with the required properties four colours. Fig. 14 shows an example of such a dissection of a 4×3 chess-board (an elementary consideration will convince the reader that three colours cannot suffice in this case).

Hence it is evident that for the case $n = 2$ the necessary number of 4 colours generally cannot be reduced. It is not known to the author whether the number $2n$ of colours, sufficient according to theorem 9, can be reduced. The author is not acquainted even with such a dissection of a three-dimensional chess-board S into $1 \times 1 \times 2$ rectangular parallelepipeds that, when coloured, requires 6 colours. It can easily be proved that: if we need for the colouring of the $m \times n \times p$ chess-board S with the dissection $\bar{D}$ 6 colours, all the three numbers m, n, p must be greater than 2. The reader can see at once that to colour the linear factor of the graph $G(2, 3, 3)$ given in fig. 11, he must have 5 colours. Whence it follows: if the number of necessary colours in colouring a three-dimensional chess-board can be reduced, this lower limit will generally not be less than 5. Even in this dissection of the $2 \times 3 \times 3$ chess-board into $1 \times 1 \times 2$ rectangles that corresponds to the linear factor from fig. 12, we require for the colouring of the considered chess-board 5 colours.

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О ЛИНЕЙНЫХ ФАКТОРАХ В РЕШЕТЧАТЫХ ГРАФАХ

Антон Коциг

Резюме

Пусть $\xi_1, \xi_2, \ldots, \xi_n$ — целые числа >1, n — целое положительное число. Под решетчатым графом $G(\xi_1, \xi_2, \ldots, \xi_n)$ мы будем понимать граф, в котором: (а) множество вершин образовано множеством V точек эвклидова пространства E_n, определенным следующим образом: точка x с координатами $x_1, x_2, \ldots, x_n$ принадлежит V тогда и только тогда, когда для всех $i = 1, 2, \ldots, n$ выполняется: x_i есть целое положительное число $\leqq \xi_i$; (б) две вершины из V соединены ребром (причем единственным ребром) тогда и только тогда, когда их расстояние равно 1. Оси координат в E_n мы будем обозначать через $X_1, X_2, \ldots, X_n$. Если ребро h соединяет в $G(\xi_1, \xi_2, \ldots, \xi_n)$ вершины $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, причем $x_i \neq y_i$, то будем говорить, что ребро h параллельно X_i. Пусть i — произвольное число из $\{1, 2, \ldots, n\}$, Если из графа $G(\xi_1, \xi_2, \ldots, \xi_n)$ удалить все ребра, параллельные X_i, то получится граф. имеющий ξ_i компонентов. Эти компоненты мы будем называть слоями в направлении оси X_i; термин k-ый слой в направлении оси X_i мы будем употреблять для слоя, содержащего вершину $x = (x_1, x_2, \ldots, x_n)$, в которой $x_i = k$, $x_j = 1$ для всех $i \in \{1, 2, \ldots, n\}; j \neq i$.

Линейный фактор L решетчатого графа мы будем называть существенным линейным фактором, если для всех $i \in \{1, 2, \ldots, n\}$ выполняется: для всяких двух соседних слоев в направлении оси X_i существует хотя бы одно такое ребро из L, которое соединяет вершину из одного слоя с вершиной из второго слоя.

Пусть ребро g соединяет в решетчатом графе вершины x, y, а ребро $\dot{g}$ — вершины $\dot{x}$, $\dot{y}$. Будем говорить, что ребра g, $\dot{g}$ близки, если выполняется: (1) $\{x, y\} \cap \{\dot{x}, \dot{y}\} = \emptyset$; (2) в графе существует четырехугольник, содержащий ребра g, $\dot{g}$. Оставшиеся два ребра указанного четырехугольника назовем поперечниками близких ребер g, $\dot{g}$. Пусть L — произвольный линейный фактор решетчатого графа, содержащий два его близких ребра g, $\dot{g}$, и пусть L^* — подграф того же решетчатого графа, который получится из L, если в последнем заменить ребра g, $\dot{g}$ их поперечниками; тогда L^* также является линейным фактором рассматриваемого графа. Будем также говорить, что L^* получается $\varkappa$-преобразованием L на ребрах g, $\dot{g}$.

Под бесконечным n-мерным решетчатым графом (обозначение $G[n]$) мы будем понимать граф, вершинами которого являются все точки из E_n, все координаты которых суть целые числа, причем произвольные две вершины соединены ребром опять тогда и только тогда, когда их расстояние равно 1. О частичном графе F графа $G[n]$ будем говорить, что он является Λ-графом в $G[n]$ если справедливо: произвольная вершина из $G[n]$ инцидентна по большей мере с одним ребром из F. Пусть g, h — произвольные два ребра из $G[n]$ и пусть ребро g соединяет вершины v, w, а ребро h — вершины x, y. Будем говорить, что ребра g, h — соседние, если $g \neq h$ и если $\{v, w\} \cap \{x, y\} \neq \emptyset$; ребра g, h — почти соседние, если $g \neq h$, g и h — не соседние и если в $G[n]$ существует ребро f такое, что f, g — соседние, а также f, h — соседние ребра.

Будем говорить, что Λ — граф F графа $G[n]$ можно раскрасить p цветами (p — натуральное число), если всякому ребру из F можно поставить в соответствие число (= цвет) из $\{1, 2, \ldots, n\}$ так, чтобы произвольные два почти соседние ребра были окрашены в разный цвет.

В работе доказываются следующие теоремы:

1. В решетчатом графе $G(\xi_1, \xi_2, \ldots, \xi_n)$ с четным числом вершин существует n и только n таких линейных факторов, никакие два из которых не имеют общего ребра.

2. Пусть L — произвольный линейный фактор решетчатого графа $G(\xi_1, \xi_2, \ldots, \xi_n)$. Для числа $\varrho_k(i)$ ребер из L, соединяющих некоторую вершину из k-ого слоя с некоторой вершиной из $(k+1)$-ого слоя в направлении оси X_i, выполняется:

$$k \prod_{\substack{j=1 \\ j \neq i}}^{n} \xi_j \equiv \varrho_k(i) \pmod 2$$

3. В графе $G(\xi_1, \xi_2)$ существует существенный линейный фактор тогда и только тогда, когда (1) $\xi_1 \xi_2 \equiv 0 \pmod 2$; (2) $\xi_1 \geqq 5$; (3) $\xi_2 \geqq 5$; (4) не имеет места $\xi_1 = \xi_2 = 6$.

4. В трехмерном решетчатом графе $G(\xi_1, \xi_2, \xi_3)$ существует существенный линейный фактор тогда и только тогда, когда он имеет четное число вершин и когда хотя бы две из чисел ξ_1, ξ_2, ξ_3 больше чем 2.

5. Пусть $n > 3$. В n-мерном решетчатом графе существует существенный линейный фактор тогда и только тогда, когда он имеет четное число вершин.

6. Произвольный линейный фактор двухмерного решетчатого графа $G(\xi_1, \xi_2)$ содержит хотя бы одну его пару близких ребер и произвольный линейный фактор может быть некоторым конечным числом $\varkappa$-преобразований переведен в произвольный другой линейный фактор того же графа.

7. Пусть p — произвольное натуральное число. Решетчатый граф $G(\xi_1, \xi_2, \ldots, \xi_{4p})$, в котором $\xi_i = 2$ для всех $i \in \{1, 2, \ldots, 4p\}$, можно разложить на $4p$ линейных факторов так, что никакой из линейных факторов этого разложения не будет содержать пары близких ребер (т. е. ребра произвольного четырехугольника из $G(\xi_1, \xi_2, \ldots, \xi_{4p})$ принадлежат четырем отличным друг от друга линейным факторам этого разложения).

8. Произвольный Λ-граф в графе $G[n]$ можно раскрасить $2n$ цветами.

9. Всякий линейный фактор n-мерного решетчатого графа может быть раскрашен $2n$ цветами.

10. Пусть S — произвольная двухмерная шахматная доска и пусть R — произвольное ее разложение на прямоугольники размеров 1×2. Прямоугольники из R можно всегда с помощью четырех цветов раскрасить так, чтобы всякие два прямоугольника, общую границу которых образует отрезок длиной $\geqq 1$, были окрашены в разный цвет.