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INTRODUCING AN ORIENTATION INTO A GIVEN NON-DIRECTED GRAPH

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In [2] A. Kotzig publishes the following problem (Problem 24, p. 162)

Let G be a non-directed graph without loops and multiple edges, with the set V of vertices. Characterize those subsets X (respectively Y) of V which arise from an orientation of G as the set of those vertices at which there is no incoming (respectively outgoing) edge.

This problem is being solved in this paper (we consider only finite graphs).

We shall assume that the graph G is connected. If it is not, we may consider each component separately. By a tree we understand any connected graph without circuits, therefore also the graph consisting of a unique isolated vertex. The symbol Γu, where u is a vertex of the graph G will signify the set of vertices of the graph G which are joined by an edge with the vertex u. If $M \subset V$, then $\Gamma M = \bigcup_{u \in M} \Gamma u$.

Theorem 1. *Let G be a non-directed graph without loops and multiple edges, its vertex set be V. The system $\mathcal{M}(G)$ (resp. $\mathcal{N}(G)$) of those subsets X (resp. Y) of the set V which arise from an orientation of G as the set of those vertices at which there is no incoming (resp. outgoing) edge is equal to the system of all internally stable (see [1]) subsets of the set V in the case where G is not a tree and is equal to the system of all non-empty internally stable subsets of the set V in the case where G is a tree.*

Before proving this theorem, we shall state some lemmas.

Lemma 1. *There is always $\mathcal{M}(G) = \mathcal{N}(G)$.*

Proof. Let $M \in \mathcal{M}(G)$ and let the graph G be directed so that the set M is the set of exactly all vertices at which there is no incoming edge. Now if we shall reverse the orientation of all edges of the graph G, the set M is evidently the set of exactly all vertices at which there is no outgoing edge. So $M \in \mathcal{N}(G)$. Analogously we can prove that $M \in \mathcal{N}(G)$ implies $M \in \mathcal{M}(G)$.

Lemma 2. *If G is a tree, then at an arbitrary orientation of G there exists at least one vertex of the graph G at which there is no incoming edge of G, so $\emptyset \notin \mathcal{M}(G)$.*

Proof. Each edge of the graph G comes exactly into one vertex. Therefore the number of the vertices at which there is at least one incoming edge cannot exceed the number of edges of the graph. As the number of edges of a tree is less than the number of its vertices, there must exist at least one vertex in G at which there is no incoming edge of G.

Lemma 3. *If G is a tree, u an arbitrary vertex of it, then G can be directed so that u is the unique vertex in G at which there is no incoming edge.*

This assertion is well-known and intuitive, so we shall not give the proof. Let us note only that the graph G thus directed is properly an „arborescence" (see [1]) with the root u.

Lemma 4. *If G is not a tree, then it can be directed so that at each vertex of G there is at least one incoming edge of G, therefore $\emptyset \in \mathscr{M}(G)$.*

Proof. Let K be an arbitrary skeleton of the graph G. Choose a vertex u of the graph G which is incident at least with one edge not belonging to K. Such a vertex exists, because G is not a tree. Direct the graph K so that u is the unique vertex in K at which there is no incoming edge of K (see Lemma 3). Now choose an edge h incident with u which does not belong to K and direct it so that it might come into u. The remaining edges of the graph G may be directed arbitrarily. In the case of such an orientation there is evidently at each vertex of the graph G some incoming edge of G.

Proof of the Theorem 1. If $M \in \mathscr{M}(G)$, then M must be internally stable. If M contained two vertices x, y joined to one another by the edge h, then at every orientation the edge h would have to come either into x or into y, and therefore either x or y would not belong to M, which is a contradiction. Now let M be an internally stable subset of the set V. The case $M = \emptyset$ has been investigated in the Lemmas 2 and 4, so assume $M \neq \emptyset$ and do not distinguish whether G is a tree or not. All edges incident with any vertex of M will be directed so that they might go out of that vertex. Edges joining two vertices of ΓM may be directed arbitrarily. Now denote by G' the graph which arises from G by removing the vertex set $M \cup \Gamma M$ and all edges incident with any vertex of that set. If G' is an empty graph, the proof is finished. If G' is non-empty, let C be a component of it. If C is not a tree, direct it so that at each vertex of C there is at least one incoming edge of C (see Lemma 4). Edges joining vertices of C with vertices of ΓM may be directed arbitrarily. If C is a tree, let u be a vertex of the graph C which is joined by an edge k with some vertex of ΓM. Such a vertex must exist, because G is connected. Direct the graph C so that u might be a unique vertex in C at which there is no incoming edge of C (see Lemma 3). Direct the edge k so that it might come into u. Other edges joining vertices of C with vertices of ΓM may be directed arbitrarily. The graph G which is thus directed has evidently the following property: The set M is the set of exactly all vertices of G at which there is no incoming edge of G. So $M \in \mathscr{M}(G)$. This was the proof of the assertion for $\mathscr{M}(G)$ and according to Lemma 1 the same must hold also for $\mathscr{N}(G)$.

II.

In this paragraph we shall investigate simultaneously the sets X and Y and consider relations between them.

Lemma 5. *Let G be a tree. Given an arbitrary decomposition of the set of its end vertices into two non-empty disjoint sets X_0, Y_0, the graph G can be directed so that $X = X_0$, $Y = Y_0$ might hold.*

Proof. If G consists of only one isolated vertex, the above mentioned decomposition does not exist. So we shall investigate only graphs which contain at least one edge and use the mathematical induction with respect to the number of edges.

Let G contain a unique edge h and its end vertices u, v. Both these vertices are the end vertices of the graph G. There exist exactly two decompositions which fulfill the condition of the lemma, they are $X_0 = \{u\}$, $Y_0 = \{v\}$ and $X_0 = \{v\}$, $Y_0 = \{u\}$. The orientation of the edge h from u into v corresponds to the first decomposition, the orientation of the edge h from v into u corresponds to the second one.

Now assume that the lemma holds for all trees containing $n - 1$ edges, where $n \geq 2$ is a positive integer. Let G be a tree with n edges and let the sets X_0, Y_0 fulfilling the condition of the lemma be given. Assume that X_0 has the cardinality greater or equal to the cardinality of Y_0. Choose a vertex $u \in X_0$. Let the edge incident with u be denoted by h, let the vertex incident with h different from u be denoted by v. Let G' be the graph which arises from the graph G by removing the vertex u and the edge h. The graph G' is evidently again a tree. Now two cases can occur: either v is an end vertex of G', or not. If v is an end vertex of G', take the decomposition of the set of the set of the end vertices of the graph G' into the sets X'_0, Y'_0, where $X'_0 = (X_0 - \{u\}) \cup \{v\}$, $Y'_0 = Y_0$.

The sets X'_0, Y'_0 are evidently non-empty and disjoint and form a decomposition of the set of end vertices of the graph G', the graph G' contains $n - 1$ edges, and so G' can be directed so that X'_0 (resp. Y''_0) might be the set of all vertices, at which there is no incoming (resp. outgoing) edge. Now if the graph G is directed so that the edge h is directed from u into v and other edges are directed in the same way as in G', an orientation of the graph G fulfilling the assertion of the lemma is obtained. If v is not an end vertex of the graph G', the graph G' is not a path and therefore contains at least three end vertices.

Decompose the set of end vertices of the graph G' into the sets X'_0, Y'_0, where $X'_0 = X_0 \dot{-} \{u\}$, $Y'_0 = Y_0$. The sets X'_0, Y'_0 are again non-empty, disjoint and form a decomposition of the set of end vertices of the graph G'. We proceed in further quite the same way as in the preceding case. In the case where the cardinality of the set X_0 is less than the cardinality of the set Y_0, we choose $u \in Y_0$ and proceed analogously.

Lemma 6. *Let G be a connected graph which is not a tree. Given an arbitrary decomposition of the set of its end vertices into two disjoint subsets X_0, Y_0, the graph G can be directed so that $X = X_0$, $Y = Y_0$.*

Here the case where one of the sets X_0, Y_0 or even both are empty is not excluded.

Proof. Denote by F the set of end vertices of the graph G. First consider the case when both the sets X_0, Y_0 are non-empty. Choose an arbitrary skeleton K of the graph G. The set of end vertices of the skeleton K will be denoted F', evidently $F \subset F'$. Now the decomposition of the set F' into two disjoint non-empty subsets X'_0, Y'_0 will be described. All vertices of the set X_0 will belong to X'_0, all vertices of the set Y_0 will belong to Y'_0. A vertex of $F' \dot{-} F$ which is joined in G with some vertex not belonging to F by an edge not belonging to K can be put into an arbitrary set of X'_0, Y'_0 .Now take the subgraph H of the graph G generated by the sets of those vertices of $F' \dot{-} F$ which are not joined with any vertex not belonging to F' by an edge not belonging to K. If some component of the graph H is a tree, choose one vertex in it and put it into Y'_0, all other vertices of that component will be put into X'_0. All vertices of the component of the graph H which is not a tree will be put into X'_0. Now direct the skeleton K so that X'_0 (resp. Y'_0) might be the set of all vertices at which there is no incoming (resp. outgoing) edge of K (see Lemma 5). If the vertex $u \in F' \dot{-} F$ is joined with a vertex $v \notin F'$ by an edge h not belonging to K and if $u \in X'_0$ (resp. $u \in Y'_0$), direct the edge h from v into u (resp. from u into v). Now let H_0 be a component of the graph H. If H_0 is a tree and w is a vertex of H_0 belonging to Y'_0, direct H_0 so that w might be the unique vertex in H_0, at which there is no incoming edge of H_0 (see Lemma 3). If H_0 is not a tree, direct it so that at every vertex of H_0 there might be at least one incoming edge of H_0 (see Lemma 4). Other edges of the graph G can be directed arbitrarily. Evidently an orientation fulfilling the assertion of the lemma is obtained.

Now let $X_0 = \emptyset$, $Y_0 \neq \emptyset$. Choose again a skeleton K of the graph G. If there exist end vertices of the skeleton K which are not end vertices of the graph G, we proceed as in the preceeding case and take $X'_0 \neq \emptyset$ which can evidently be chosen in such a way (the component of H_0 cannot be an isolated vertex, because such a vertex would be an end vertex of the graph, which would be a contradiction with the assumption that all vertices of the graph H belong to $F' - F$). If all end vertices of the skeleton K are end vertices of the graph G, choose a vertex u which is incident with an edge h of the graph G which does not belong to K. Direct the skeleton K as an „arborescence" with the root u (see Lemma 3). Further direct the edge h so that it might come into u. The remaining edges may be directed arbitrarily. Evidently the desired orientation is obtained.

In the case $X_0 \neq \emptyset$, $Y_0 = \emptyset$ we proceed analogously.

If $X_0 = Y_0 = \emptyset$, proceed again as in the first case, but $X_0' \neq \emptyset$, $Y_0' \neq \emptyset$ must hold. This can be obtained. (The set F' contains at least two vertices.) If the graph H is non-empty and contains a component which is not a tree, the assertion is evident. If H contains only components which are not trees, direct one of them according to Lemma 4. Choose an edge k in it and place its beginning vertex into Y_0', others into X_0'. If H is empty, then every vertex of F' is joined with a vertex not belonging to F' by an edge not belonging to K and can be put into an arbitrary edge of the sets X_0', Y_0'. As F' contains at least two vertices, the assertion holds also here.

Now we can express a theorem.

Theorem 2. *Let there be given a connected graph G containing at least two vertices and given two subsets X, Y of its vertex set. The graph G can be directed so that X (resp. Y) might be the set of all vertices of the graph G, at which there is no incoming (resp. outgoing) edge, if and only if the following conditions are fulfilled:*

(1) *X and Y are internally stable.*

(2) $X \cap Y = \emptyset$.

(3) *Every end vertex of the graph G belongs either to X, or to Y.*

(4) *If H_0 is a component of the graph H arisen from the graph G by removing the the set $X \cup Y$ and if H_0 is a tree, then H_0 contains a vertex u joined in G with a vertex of X and a vertex v joined in G with a vertex of Y.*

Proof. Necessity of the condition (1) is implied by Theorem 1. The intersection $X \cap Y$ at an orientation of the graph G fulfilling the assertion of the theorem would be the set of vertices, at which there are neither incoming, nor outgoing edges, i. e. the set of isolated vertices. As G is connected and contains at least two vertices, this set is empty. Therefore the condition (2) is necessary. The necessity of the condition (3) is also evident. An end vertex of the graph G is incident only with one edge and this edge cannot be at the same time an incoming edge and an outgoing one. Assume that there exists a component H_0 of the graph H which is a tree and no one of whose vertices is joined with a vertex of X. So if the vertex u of H_0 is incident with an edge h not belonging to H_0, the edge h joins the vertex u with a vertex $v \in Y$ and therefore the edge h at the orientation fulfilling the condition of the theorem must be directed from u into v. Therefore at no vertex of H_0 there is an incoming edge not belonging to H_0. But according to Lemma 2 at every orientation there must exist a vertex in H_0 at which there is no incoming edge of H_0. Therefore H_0 contains a vertex w at which there is no incoming edge of G and so $w \in X$, which is a contradiction with the assumption that w belongs to H. In a case where a vertex of H_0 is joined with a vertex of Y, the proof is analogous. Thus also the conditon (4) is necessary.

Now let the conditions (1), (2), (3), (4) be fulfilled. All edges incident with any vertex of X (resp. of Y) will be directed so that they might go out from (resp come into) that vertex. Now let H_1 be a component of the graph H. All end vertices of the graph H_1 are joined either with some vertex of X, or with some vertex of Y. If an end vertex of the graph H_1 were joined neither with a vertex of X, nor with a vertex of Y, it would be an end vertex also in the graph G and would have to belong, according to the condition (3), to X or to Y, which would be a contradiction. Decompose the set of end vertices of the graph H_1 into two disjoint subsets X_0, Y_0 so that end vertices of the graph H_1 which are joined with vertices of X (resp. of Y) and are not joined with vertices of Y (resp. of X) might belong to X_0 (resp. to Y_0), and vertices of the graph H_1 which are joined at the same time with vertices of X and of Y may belong to one arbitrary set of X_0, Y_0. If possible, we do it so that X_0 and Y_0 might be non-empty. If it is not possible, it signifies that either no end vertex of H_1 is joined with a vertex of X, or no end vertex of H_1 is joined with a vertex of Y. If H_1 is not a tree, some of the sets X_0, Y_0 may be empty. If H_1 is a tree and no end vertex of H_1 is joined with a vertex of X (resp. of Y), according to the condition (4) some internal vertex u_0 of H_1 must be joined with a vertex of X (resp. of Y), denote this case by (α) (resp. by (β)).

If neither (α), nor (β) occurs, direct the graph H_1 so that X_0 (resp. Y_0) might be the set of vertices at which there is no incoming (resp. outgoing) edge of H_1. In the case (α) direct the graph H_1 as an „arborescence" with the root u_0. In the case (β) proceed analogously as in the case (α). Do this with each component of the graph H. The orientation obtained so fulfills evidently the assertion of the theorem.

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