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ON EVEN REGULAR GRAPHS OF THE THIRD DEGREE

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In the present paper we mean by „graph" a finite, non-oriented graph without loops.

Let G be any even regular graph of the third degree without multiple edges. It is known that the number of vertices of a regular graph of the third degree is always even (see, e. g. König, [3], p. 21, theorem 3). Let $2m$ be the number of vertices of G. As G is an even graph and without multiple edges, it necessarily follows that $2m \geqq 6$. Let G' or G'' respectively be the graph arising from G by the splitting of its edge h that joins the vertices u, v (see Fig. 1 the concept of the splitting of edges was originated by Frink [2]).

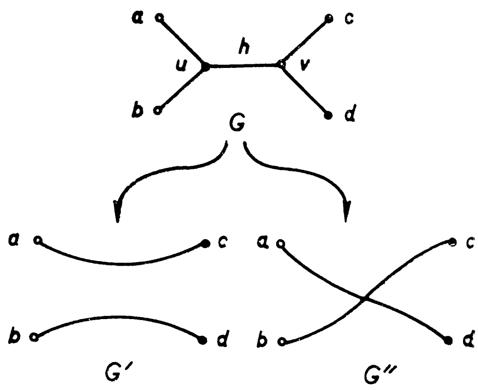

Fig. 1.

Note 1. An even graph is also called bichromatic graph (see Berge, [1], p. 30). The name was adopted because of the fact that the vertices of such a graph can always be coloured by two colours in such a way that any edge joins two vertices of different colours. In the figures we shall make use of this possibility so that the vertices of one colour will be marked by full circles, the others by void circles.

The following is evident: Any edge of an even graph without multiple edges may be split in two ways; by both ways we always get an even regular graph of the third degree. We shall say that the edge h of the graph G is X-reducible, if at least one of the graphs arising from the splitting of h does not contain multiple edges (in the reverse case we shall say that the edge h is X-irreducible). Such a splitting of the edge h, where there arises from the graph G a graph without multiple edges, will be called the X-reduction of the graph G on the edge h.

Lemma 1. *Let G be an even regular graph of the third degree without multiple edges. Any edge h of it is X-irreducible if and only if it belongs to at least two different quadrangles of G that have — apart from h — at least another edge in common. This common edge is always adjacent to h.*

Proof. Let h be any edge of the graph G and let h join the vertices u, v. The vertices joined in G by an edge of the vertex u, or v respectively, will be denoted as seen in Fig. 1. It is evident that a, b, c, d are four different vertices. There may exist such quadrangles in G that by proceeding along them we pass through the vertices of the graph in the following order (see Fig. 1): quadrangle Q_1: u, v, c, a; quadrangle Q_2: u, v, c, b; quandrangle Q_3: u, v, d, a; quadrangle Q_4: u, v, d, b. If there existed in G from among the four considered quadrangles only the quadrangles Q_1, Q_4 that — apart from h have no other edge in common —(or if only one of them existed) — then the graph G'' would not contain a multiple edge and the edge h would be X-reducible. Similarly, if in G there existed only the quadrangles Q_2, Q_3 (from among the four considered quadrangles), then the graph G' would not contain multiple edges and the edge h would be X-reducible. If both the graph G' and the graph G'' are to contain multiple edges, there must exist in G at least one of the following four pairs of quadrangles: $\{Q_1, Q_2\}$, $\{Q_1, Q_3\}$, $\{Q_2, Q_4\}$, $\{Q_3, Q_4\}$. Each of the mentioned four pairs of quadrangles has the following property: the quandrangles of the pair have — apart from h — another edge in common and this edge is adjacent to h. The lemma is proved.

We shall say that en even regular graph of the third degree is X-irreducible if each of its edges is X-irreducible. The graph in Fig. 2 is an example of such a graph. It will be denoted further by the symbol G^*.

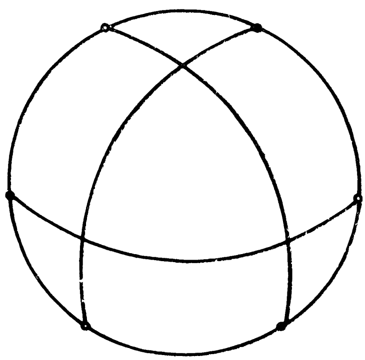

Fig. 2.

Theorem 1. *Each component of an X-irreducible graph is isomorphic with the graph G^*.*

Proof. It is evident that the component of an X-irreducible graph is an X-irreducible graph. It suffices therefore to prove that each connected X-irreducible graph is isomorphic to G^*.

Let G be a connected X-irreducible graph and let h be any of its edges; let u, v be vertices incident at h. The edge h is X-irreducible. According to lemma 1, the edge h belongs to two different quadrangles Q, Q', which, apart from h, have another edge in common (let us denote it by g), and g, h are adjacent. We can assume without loss of generality that the edge g joins the vertex v with a certain vertex w. It is evident that the quadrangles Q, Q' cannot have three edges in common (as the fourth edge from Q and the fourth edge from Q' would be multiple edges, which is impossible in a X-irreducible graph). It follows that the graph G contains as a partial subraph the graph F, given in Fig. 3, where r, s are two different vertices.

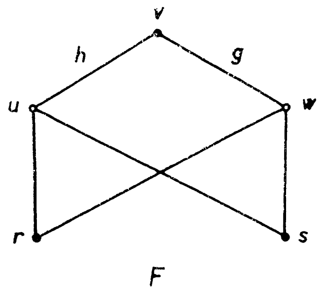

Fig. 3.

Let us denote by the symbol f_r (or f_v, or f_s, respectively) that edge incident at the vertex r (or v, or s, respectively) that does not belong to F and let $\bar{r}$ (or $\bar{v}$, or $\bar{s}$, respectively) be the second vertex, at which the edge f_r (or f_v, or f_s, respectively) is incident. The edge f_r is X-irreducible. Therefore f_r must belong to a certain quadrangle Q''. This quadrangle contains r and must therefore contain further either the edge joining r with u, or the edge joining r with w. Hence: either $\bar{r} = \bar{v}$, or $\bar{r} = \bar{s}$. By a similar consideration with respect to the edge f_v and f_s we find that $\bar{r} = \bar{v} = \bar{s}$ holds. But then the graph F with the edges f_r, f_v, f_s and with the vertex $\bar{r} = \bar{v} = \bar{s}$ forms a graph that represents the whole graph G and G is isomorphic to G^*. This proves the theorem.

Let G be any even regular graph of the third degree without multiple edges and let $R = \{V_0, V_1\}$ be such a decomposition of the set of vertices of the graph G that the following holds: any edge from G joins vertices from different classes of the decomposition R.

Note 2. It is known that if n is the number of components of the even graph G and m is the number of decompositions with the above property, then $m = 2^{n-1}$, hence in an even graph there exists at least one such decomposition.

Let $u_0 \neq v_0$ be vertices from V_0 and $u_1 \neq v_1$ vertices from V_1 such that in the graph G there exists an edge g joining the vertices u_0, u_1 and there exists the edge h joining the vertices v_0, v_1. If the graph G' arises from G in the way shown in Fig. 4, we shall say that G' arose from G by an X-extension on the edges g, h. Hence X-extension is the inverse process of X-reduction.

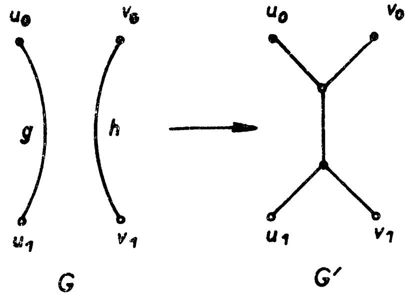

Fig. 4.

Lemma 2. *Let G be any even regular graph of the third degree without multiple edges and let g, h be two of its edges that are not adjacent. At the edges g, h exactly one X-extension is possible and the graph G' which arises in this way is always an even regular graph of the third degree without multiple edges.*

The proof is evident.

Note 3. The X-extension is defined only on pairs of edges that are not adjacent. If we extended the graph G in a similar way on two adjacent edges, then the graph G' would contain multiple edges. (see Fig. 4).

Theorem 2. *Any even regular graph of the third degree with $2n$ vertices without multiple edges is either X-irreducible, or it may be constructed by repeated X-extensions from a certain X-irreducible graph with m components, which are all isomorphic to the graph G^*, where $6m \leq 2n$.*

Proof. Theorem 2 is a direct consequence of theorem 1, of the definition of the X-irreducible graph and of the relation between X-reduction and X-extension.

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