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INDUCTIVE TENSOR PRODUCT OF VECTOR-VALUED MEASURES

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The aim of this note is to prove the following proposition. Let measurable spaces $(S, \mathscr{S})$ and $(T, \mathscr{T})$, locally convex topological vector spaces X and Y, and (σ-additive) vector-valued measures $\mu : \mathscr{S} \to X$ and $\nu : \mathscr{T} \to Y$ be given. If we denote by $\mathscr{S} \otimes_\sigma \mathscr{T}$ the σ-ring generated by the sets of the form $E \times F$, $E \in \mathscr{S}$, $F \in \mathscr{T}$, and by $X \check{\otimes} Y$ the (completed) inductive tensor product of the spaces X and Y, then there exists a unique vector-valued measure $\lambda :$ $: \mathscr{S} \otimes_\sigma \mathscr{T} \to X \check{\otimes} Y$ such that the relation

$$(1) \qquad \lambda(E \times F) = \mu(E) \otimes \nu(F), E \in \mathscr{S}, F \in \mathscr{T}$$

holds.

Let $\mathscr{S}$ be a σ-algebra of subsets of a set S. We denote by $ca\ (\mathscr{S})$ the Banach space of all (finite) complex-valued (σ-additive) measures on $\mathscr{S}$, and for any $m \in ca\ (\mathscr{S})$ let $\|m\| = |m|\ (S)$, where $|m|$ is the variation of measure m.

In the following, key use will be made of the following result, which can be of some interest in other connections too.

Lemma. *Let $\mathscr{S}$ and $\mathscr{T}$ be σ-algebras. Let $\{m_\alpha\}_{\alpha \in A} \subset ca\ (\mathscr{S})$ be a bounded set and let the measures m_α, $\alpha \in A$, be uniformly absolutely continuous with respect to $m \in ca\ (\mathscr{S})$, $m \geqq 0$. Let $\{n_\beta\}_{\beta \in B} \subset ca\ (\mathscr{T})$ be a bounded set and let n_β, $\beta \in B$, be uniformly absolutely continuous with respect to $n \in ca\ (\mathscr{T})$, $n \geqq 0$. Then the measures $m_\alpha \times n_\beta$, $(\alpha, \beta) \in A \times B$, are uniformly absolutely continuous with respect to $m \times n$.*

Proof. Let $\|m_\alpha\| \leqq K_1$ for $\alpha \in A$ and let $\|n_\beta\| \leqq K_2$, $\beta \in B$. Then $\|m_\alpha \times$ $\times\, n_\beta\| \leqq K_1 K_2$ for every pair $(\alpha, \beta) \in A \times B$. Thus the set of the measures $m_\alpha \times n_\beta$ for $(\alpha, \beta) \in A \times B$ forms a bounded subset in $ca\ (\mathscr{S} \otimes_\sigma \mathscr{T})$.

Let now $\{E_i\}$ be a monotone decreasing sequence of sets in $\mathscr{S} \otimes_\sigma \mathscr{T}$, and let $\bigcap_{i=1}^{\infty} E_i = \emptyset$. We prove that $\lim_{i \to \infty} m_\alpha \times n_\beta\ (E_i) = 0$ uniformly with respect to $(\alpha, \beta) \in A \times B$. Let $\varepsilon > 0$. Take $\delta_1 > 0$ such that for $F \in \mathscr{S}$, $m(F) <$ $< \delta_1$, we have $|m_\alpha(F)| < \varepsilon$ for all $\alpha \in A$. For every $s \in S$ we have $\bigcap_{i=1}^{\infty} (E_i)_s =$

$= \emptyset$ ($E_s = \{t : (s, t) \in E\}$, see [3; §34]). Thus the sequence of functions $f_i(s) = n((E_i)_s)$ converges to 0 for each $s \in S$. By the Egoroff's theorem there exists a set $F \in \mathscr{S}$ such that $m(F) < \delta_1$ and on $S - F$ $\lim_{i\to\infty} f_i(s) = 0$ uniformly with respect to s. Choose $\delta_2 > 0$ such that for $G \in \mathscr{T}$, $n(G) < \delta_2$, we have $|n_\beta(G)| < \varepsilon$ for all $\beta \in B$. Let i_0 be such a number that for $i > i_0$ we have $|f_i(s)| < \delta_2$ for all $s \in S - F$, hence $|n_\beta((E_i)_s)| < \varepsilon$ for $i > i_0$ and $s \in S - F$. Then for $i > i_0$ and $(\alpha, \beta) \in A \times B$ we have

$$|m_\alpha \times n_\beta (E_i)| = \left| \int_S n_\beta ((E_i)_s)\, \mathrm{d}m_\alpha (s) \right| \leqq \left| \int_F n_\beta ((E_i)_s)\, \mathrm{d}m_\alpha(s) \right| +$$

$$+ \left| \int_{S-F} n_\beta ((E_i)_s)\, \mathrm{d}m_\alpha (s) \right| \leqq \int_F |n_\beta ((E_i)_s)|\, \mathrm{d}|m_\alpha|(s) + \int_{S-F} |n_\beta((E_i)_s)|\, \mathrm{d}|m_\alpha|\, (s) \leqq$$

$$\leqq K_2 |m_\alpha|\, (F) + \varepsilon |m_\alpha| (S - F) \leqq 4\, K_2 \varepsilon + \varepsilon K_1.$$

It follows that the set of the measures $\{m_\alpha \times n_\beta\}$, $(\alpha, \beta) \in A \times B$ is uniformly σ-additive. In view of [1; IV.9.1] we have that it is weakly relatively compact in $ca(\mathscr{S} \otimes_\sigma \mathscr{T})$ and in view of [1; IV.9.2] there exists a measure $p \in ca$ $(\mathscr{S} \otimes_\sigma \mathscr{T})$, $p \geqq 0$, such that the measures $m_\alpha \times n_\beta$ are uniformly absolutely continuous with respect to p. Furthermore, p can be chosen so that $p(E) \leqq \sup \{|m_\alpha \times n_\beta|\, (E) : (\alpha, \beta) \in A \times B\}$ for every $E \in \mathscr{S} \otimes_\sigma \mathscr{T}$ [1; IV.9.3] (see also [6; Theorem 3.10]). Let p be chosen in such a manner. As every measure $m_\alpha \times n_\beta$ is absolutely continuous with respect to $m \times n$, the equality $m \times n(E) = 0$ implies $p(E) = 0$. Thus p is absolutely continuous with respect to $m \times n$. Hence the measures $m_\alpha \times n_\beta$, $(\alpha, \beta) \in A \times B$ are uniformly absolutely continuous with respect to $m \times n$.

Corollary. *If $\{m_\alpha\}_{\alpha \in A}$ is a weakly relatively compact subset in ca $(\mathscr{S})$ and $\{n_\beta\}_{\beta \in B}$ is a weakly relatively compact subset in ca $(\mathscr{T})$, then $\{m_\alpha \times n_\beta\}$, $(\alpha, \beta) \in A \times B$ is a weakly relatively compact subset in ca $(\mathscr{S} \otimes_\sigma \mathscr{T})$.*

Proof. By [1; IV.9.2] a set $M \subset ca$ $(\mathscr{S})$ is weakly relatively compact if and only if it is bounded and there exists a measure $m \in ca$ $(\mathscr{S})$, $m \geqq 0$, such that the measures in M are uniformly absolutely continuous with respect to m.

Let now X and Y be locally convex spaces. Let the topology of the space X be determined by a system of seminorms $\{\|\ \|_\alpha\}_{\alpha \in A}$ and let the topology of the space Y be determined by a system of seminorms $\{\|\ \|_\beta\}_{\beta \in B}$. X' and Y' denote dual spaces of X and Y, respectively. For $x' \in X'$ we denote $\|x'\|_\alpha = \sup \{|\langle x, x' \rangle| : \|x\|_\alpha \leqq 1\}$ for every $\alpha \in A$. Similarly for Y.

The topology of $X \otimes Y$ determined by the system of seminorms

$$(2) \qquad \|\sum_{i=1}^{k} x_i \otimes y_i\|^{\vee}_{(\alpha,\beta)} = \sup\{|\sum_{i=1}^{k} \langle x_i, x'\rangle\langle y_i, y'\rangle| : \|x'\|_\alpha \leqq 1,\ x' \in X';\ \|y'\|_\beta \leqq 1,\ y' \in Y'\},\ (\alpha, \beta) \in A \times B$$

is called the inductive tensor topology. The completion of the space $X \otimes Y$ under this topology is the inductive tensor product $X \check{\otimes} Y$ of the spaces X and Y.

We denote by $X \hat{\otimes} Y$ the projective tensor product of the spaces X and Y. (These notions are introduced in [2]. See also [5].)

Theorem. *Let $\mathscr{S}$ and $\mathscr{T}$ be σ-algebras. Let μ: $\mathscr{S} \to X$ and ν: $\mathscr{T} \to Y$ be vector-valued measures.*

Then there exists a unique vector-valued measure λ: $\mathscr{S} \otimes_\sigma \mathscr{T} \to X \check{\otimes} Y$ such that (1) *holds.*

Proof. If a set G is of the form

$$(3) \qquad G = \bigcup_{i=1}^{k} E_i \times F_i,$$

where the union is disjoint and $E_i \in \mathscr{S}$, $F_i \in \mathscr{T}$, then in view of the additivity condition and the condition (1) we define

$$(4) \qquad \lambda(G) = \sum_{i=1}^{n} \mu(E_i) \otimes \nu(F_i).$$

It is easy to see that the function λ is unambiguously defined by the equality (4) on the algebra $\mathscr{S} \otimes \mathscr{T}$ of the sets of the form (3) and that it is additive.

We must prove that λ is σ-additive and can be extended to a σ-additive function on the σ-algebra $\mathscr{S} \otimes_\sigma \mathscr{T}$ generated by the algebra $\mathscr{S} \otimes \mathscr{T}$ with values in $X \check{\otimes} Y$. It is known (see e. g. [4; §4]) that such an extension (if it exists) is only one.

For every $\alpha \in A$ there exists a measure $m_\alpha \in \text{ca}\ (\mathscr{S})$, $m_\alpha \geqq 0$, such that $\|\mu(E)\|_\alpha \to 0$ if $m_\alpha(E) \to 0$. Similarly, for every $\beta \in B$ there exists $n_\beta \in ca\ (\mathscr{T})$, $n_\beta \geqq 0$, such that $\|\nu(F)\|_\beta \to 0$ if $n_\beta(F) \to 0$ (see [4; 4.2]).

From there it is obvious that the measures $\langle\mu(\ ), x'\rangle$ for $\|x'\|_\alpha \leqq 1$, $x' \in X'$, are uniformly absolutely continuous with respect to m_α and form a bounded subset in $ca\ (\mathscr{S})$. Similarly, the measures $\langle\nu(\ ), y'\rangle$, $\|y'\|_\beta \leqq 1$, $y' \in Y'$, are uniformly absolutely continuous with respect to n_β and form a bounded subset in $ca\ (\mathscr{T})$. By the Lemma the product measures $\langle\mu(\ ), x'\rangle \times \langle\nu(\ ), y'\rangle$ for $\|x'\|_\alpha \leqq 1$, $\|y'\|_\beta \leqq 1$, are uniformly absolutely continuous with respect to $m_\alpha \times n_\beta$. By (4) and (2) this implies immediately that $\|\lambda(G)\|^{\vee}_{(\alpha,\beta)} \to 0$ for $m_\alpha \times n_\beta(G) \to 0$. This holds for each $(\alpha, \beta) \in A \times B$.

From there it follows immediately that λ is σ-additive on $\mathscr{S} \otimes \mathscr{T}$. Further, from the proved it follows by [4; Theorem 4.2] that λ can be extended uniquely to the σ-ring $\mathscr{S} \otimes_\sigma \mathscr{T}$. ($\mathscr{S} \otimes \mathscr{T}$ is the dense subset in $\mathscr{S} \otimes_\sigma \mathscr{T}$ in the uniform structure on $\mathscr{S} \otimes_\sigma \mathscr{T}$ defined by the system of pseudometrics $\varrho_{(\alpha, \beta)}(G_1, G_2) = m_\alpha \times n_\beta(G_1 \Delta G_2)$ and λ is uniformly continuous on $\mathscr{S} \otimes \mathscr{T}$; hence it can be extended by continuity to whole $\mathscr{S} \otimes_\sigma \mathscr{T}$.)

Corollary 1. *If the space X is nuclear* ([2; 2.1] or [5; III:4.2]), *then there exists a unique vector-valued measure* λ: $\mathscr{S} \otimes_\sigma \mathscr{T} \to X \hat{\otimes} Y$ *such that* (1) *holds.*

Proof. If X is nuclear, then the projective tensor product $X \hat{\otimes} Y$ and the inductive tensor product $X \check{\otimes} Y$ coincide.

Corollary 2. *Let* $\mathscr{S}$ *and* $\mathscr{T}$ *be* σ*-rings* (δ*-rings*) *and let* $\mathscr{S} \otimes_\sigma \mathscr{T}$ ($\mathscr{S} \otimes_\delta \mathscr{T}$) *be the* σ*-ring* (δ*-ring*) *generated by the system of the sets of the form* $E \times F, E \in \mathscr{S}$, $F \in \mathscr{T}$. *Let* μ: $\mathscr{S} \to X$ *and* ν: $\mathscr{T} \to Y$ *be vector-valued measures.*

Then there exists a unique vector-valued measure λ: $\mathscr{S} \otimes_\sigma \mathscr{T}$ ($\mathscr{S} \otimes_\delta \mathscr{T}$) $\to X \check{\otimes} Y$ *for which* (1) *holds.*

Proof. Let $\mathscr{S}$ and $\mathscr{T}$ be σ-rings. For every $\alpha \in A$ and $\beta \in B$ there exist the sets $S_\alpha \in \mathscr{S}$ and $T_\beta \in \mathscr{T}$ such that $\|\mu(E - S_\alpha)\|_\alpha = 0$ for all $E \in \mathscr{S}$ and $\|\nu(F - T_\beta)\|_\beta = 0$ for all $F \in \mathscr{T}$ ([4; Theorem 3.1]). Evidently, we can now use the Theorem.

If $\mathscr{S}$ and $\mathscr{T}$ are δ-rings, then to every set $G \in \mathscr{S} \otimes_\delta \mathscr{T}$ there exist sets $E \in \mathscr{S}$ and $F \in \mathscr{T}$ such that $G \subset E \times F$. Further, the system of those sets $G \in \mathscr{S} \otimes_\delta \mathscr{T}$ for which $G \subset E \times F$ is a σ-algebra of the subsets in $E \times F$. From there we deduce the proposition as in the Theorem.

A bilinear mapping U: $X \times Y \to Z$, where Z is a locally convex space, is said to be hypercontinuous, if the linear mapping induced by it on $X \otimes Y$ is continuous under the inductive topology. The Theroem implies immediately

Corollary 3. *Let* U: $X \times Y \to Z$ *be a hypercontinuous linear mapping and let Z be a* (*sequentially*) *complete space. Let* μ: $\mathscr{S} \to X$ *and* ν: $\mathscr{T} \to Y$ *be vector-valued measures.*

Then there exists a unique vector-valued measure λ: $\mathscr{S} \otimes_\sigma \mathscr{T}(\mathscr{S} \otimes_\delta \mathscr{T}) \to Z$ *for which*

$$\lambda(E \times F) = U(\mu(E), \nu(F)), E \in \mathscr{S}, F \in \mathscr{T}.$$

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