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REMARKS ON SOME ALGEBRAIC IDENTITIES

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1. Let

$$a_i \qquad (i = 0, 1, \ldots, n)$$

and

$$x_j \qquad (j = 1, 2, \ldots, m)$$

be given complex numbers; the a_i are distinct while the x_j are arbitrary. Put

$$S(m, n) = \sum_{i=0}^{n} \frac{(a_i - x_1)(a_i - x_2) \ldots (a_i - x_m)}{(a_i - a_0) \ldots (a_i - a_{i-1})(a_i - a_{i+1}) \ldots (a_i - a_n)}. \tag{1}$$

Bartoš and Kaucký [2] have proved that

$$S(n + 1, n) = \sum_{i=0}^{n} a_i - \sum_{j=1}^{n+1} x_j, \tag{2}$$

$$S(n, n) = 1, \tag{3}$$

$$S(m, n) = 0 \qquad (m < n). \tag{4}$$

Two proofs of these formulas are given, one by induction and one using the calculus of residues.

It may be of interest to point out that these results can be obtained very rapidly by means of the Lagrange interpolation formula. We recall that if $p(x)$ denotes an arbitrary polynomial of degree $\leq n$, then

$$p(x) = \sum_{i=0}^{n} \frac{\Phi(x)}{x - a_i} \frac{p(a_i)}{\Phi'(a_i)},$$

where

$$\Phi(x) = (x - a_0)(x - a_1) \ldots (x - a_n),$$

$$\Phi'(a_i) = (a_i - a_0) \ldots (a_i - a_{i-1})(a_i - a_{i+1}) \ldots (a_i - a_n).$$

Now consider the polynomial

$$F(x) = \sum_{i=0}^{n} \frac{\Phi(x)}{x - a_i} \frac{f(a_i)}{\Phi'(a_i)}, \tag{5}$$

where

$$f(x) = (x - x_1)(x - x_2) \ldots (x - x_m).$$

It follows at once from (5) that

$$F(a_i) = f(a_i) \qquad (i = 0, 1, \ldots, n).$$

Therefore $F(x)$ is the remainder obtained when $f(x)$ is divided by $\Phi(x)$, that is

$$f(x) = Q(x)\Phi(x) + F(x), \tag{6}$$

where

$$\begin{cases} Q(x) = 0 & (m \leq n), \\ \deg Q(x) = m - n - 1 & (m > n). \end{cases}$$

If we put

$$F(x) = c_0x^n + c_1x^{n-1} + \ldots + c_n,$$

it is clear from (1) and (5) that

$$c_0 = S(m, n).$$

Hence if $m = n + 1$, $Q(x) = 1$ and we get (2); if $m = n$, $Q(x) = 0$, $c_0 = 1$; if $m < n$, $Q(x) = 0$, $c_0 = 0$.

By means of (6) we may compute c_0 for any $m > n$.

2. Bartoš and Kaucký make several applications of (2), (3) and (4). In particular, by taking

$$a_i = ai + i^2, \quad x_j = -x - j,$$

they obtain

$$\sum_{i=0}^{n} (-1)^i \frac{\binom{n}{i}^2 \binom{x + n + ai + i^2}{n}}{\binom{a + 2i - 1}{i} \cdot \binom{a + n + i}{n - i}} = (-1)^n \, n!, \tag{7}$$

$$\sum_{i=0}^{n} (-1)^i \frac{\binom{n}{i}^2 \binom{x + m + ai + i^2}{m}}{\binom{a + 2i - 1}{i} \binom{a + n + i}{n - i}} = 0 \ (0 \leq m < n). \tag{8}$$

If we put

$$(a)_k = a(a+1)\dots(a+k-1), \quad (a)_0 = 1,$$

(7) and (8) become

$$(9) \qquad \sum_{i=0}^{n} \frac{(a)_i(\frac{1}{2}a+1)_i(-n)_i}{i!(\frac{1}{2}a)_i(a+n+1)_i} \binom{x+n+ai+i^2}{n} = (-1)^n(a+1)_n,$$

$$(10) \qquad \sum_{i=0}^{n} \frac{(a)_i(\frac{1}{2}a+1)_i(-n)_i}{i!(\frac{1}{2}a)_i(a+n+1)_i} \binom{x+m+ai+i^2}{m} = 0 \ (0 \le m < n).$$

If we define

$$(11) \qquad R(n, m) = \sum_{i=0}^{n} \frac{(a)_i(\frac{1}{2}a+1)_i(-n)_i}{i!(\frac{1}{2}a)_i(a+n+1)_i} (ai+i^2)^m,$$

it is clear that (9) and (10) are equivalent to

$$(12) \qquad R(n, m) = \begin{cases} (-1)^n\, n!(a+1)_n & (m = n) \\ 0 & (0 \le m < n). \end{cases}$$

It follows at once from (11) that

$$(13) \ R_n(z) = \sum_{m=0}^{\infty} (-1)^m R(n, m) z^{-m} = \sum_{i=0}^{n} \frac{(a)_i(\frac{1}{2}a+1)_i(-n)_i}{i!(\frac{1}{2}a)_i(a+n+1)_i} \frac{z}{z+ai+i^2}.$$

If we put

$$z = \beta\gamma,\ a = \beta+\gamma,\ z+ai+i^2 = (\beta+i)(\gamma+i),$$

(13) becomes

$$(14) \qquad R_n(z) = {}_5F_4\left[\begin{matrix} a, & \frac{1}{2}a+1, & \beta, & \gamma, & -n \\ & \frac{1}{2}a, & \gamma+1, & \beta+1, & a+n+1 \end{matrix}\right]$$

in the usual notation of generalized hypergeometric series. If we recall that [1, p. 25]

$$(15) \qquad {}_5F_4\left[\begin{matrix} a, & \frac{1}{2}a+1, & c, & d, & -n \\ & \frac{1}{2}a, & a-c+1, & a-d+1, & a+n+1 \end{matrix}\right] =$$

$$= \frac{(a+1)_n(a-c-d+1)_n}{(a-c+1)_n(a-d+1)_n},$$

we find that (14) reduces to

$$R_n(z) = \frac{n!(a+1)_n}{(\gamma+1)_n(\beta+1)_n} = \frac{n!(a+1)_n}{\prod_{i=1}^{n}(z+ai+i^2)}. \tag{16}$$

We have therefore obtained a generating function for $R(n, m)$. Clearly (16) contains (12).

3. Consider next the sum

$$U(n, m) = \sum_{i=0}^{n} \frac{(a)_i(\frac{1}{2}a+1)_i(-n)_i}{i!(\frac{1}{2}a)_i(a+n+1)_i}(i+1)_m(a-m+i)_m. \tag{17}$$

It is easily verified that

$$U(n, m) = m!(a-m)_m \cdot {}_5F_4\left[\begin{matrix} a, \ \frac{1}{2}a+1, \ a, \ m+1, \ -n \\ \frac{1}{2}a, \ 1, \ a-m, \ a+n+1 \end{matrix}\right].$$

Applying (15), we get

$$U(n, m) = (-1)^n \, m! \binom{m}{n} \frac{(a-m)_m}{(a-m)_n}(a+1)_n. \tag{18}$$

In particular we have

$$U(n, m) = \begin{cases} (-1)^n n!(a+1)_n & (m = n), \\ 0 & (0 \leq m < n). \end{cases} \tag{19}$$

If we define the coefficients $A(m, k)$ by means of

$$z^m = \sum_{k=0}^{m} A(m, k) \prod_{i=1}^{k}(z+ai-i^2),$$

then

$$x^m(a+x)^m = \sum_{k=0}^{m} A(m, k) \prod_{i=1}^{k}(x+i)(a+x-i) = \sum_{k=0}^{m} A(m, k)(x+1)_k(a+x-k)_k.$$

Thus, comparing (17) and (11), we get

$$R(n, m) = \sum_{k=0}^{m} A(m, k)U(n, k). \tag{20}$$

In particular (20) implies the equivalence of (12) and (19).

REFERENCES

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[2] Bartoš P., Kaucký J., *A genesis for combinatorial identities*, Mat.-fyz. časop. *16* (1966), 31—40.

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