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# A Contribution to my Article: 'Introducing an Orientation into a Given Non-Directed Graph'

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# A CONTRIBUTION TO MY ARTICLE „INTRODUCING AN ORIENTATION INTO A GIVEN NON-DIRECTED GRAPH"

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The above mentioned paper [1] investigates only finite graphs. Here we generalize the theorems of that article for the case of infinite graphs. The Lemmas 1, 3, 4 can be proved without the assumption that the graph $G$ is finite. We generalize the Lemmas 2,5,6.

**Lemma 2a.** *If $G$ is a tree without infinite paths, then at an arbitrary orientation of $G$ there exists at least one vertex of the graph $G$ at which there is no incoming edge of $G$, so that $\emptyset \notin \mathscr{M}(G)$.*

Proof. Let a graph $G$ and its arbitrary orientation be given. Choose a vertex $u_0$ in $G$ and construct a sequence of vertices $\{u_n\}$ and a sequence of edges $\{h_n\}$ ($n$ is a positive integer) recurrently. When the vertex $u_{n-1}$ is constructed, then $h_{n-1}$ is one of the edges incoming into $u_{n-1}$ (if any). If the edge $h_{n-1}$ is constructed, then $u_n$ is its beginning vertex. Now three cases can occur:
(1) There exists a vertex $u_n$ for each non-negative integer $n$ and $u_m \neq u_n$ for $m \neq n$.
(2) There exists a vertex $u_n$ for each non-negative integer $n$ and for some $m$, $n$ there is $m \neq n$ and $u_m = u_n$.
(3) For some non-negative integer $m$ the edge $h_{m-1}$ and the vertex $u_m$ cannot be constructed, i.e. at the vertex $u_{m-1}$ there exists no incoming edge.

In case (1) vertices $u_n$ and edges $h_n$ for $n = 0, 1, \ldots$ form an infinite path in $G$. In case (2), if $m$ is the least non-negative integer such that $u_m = u_n$ for $n < m$, the vertices $u_n, u_{n+1}, \ldots, u_m$ and the edges $h_n, h_{n+1}, \ldots, h_{m-1}$ form a circuit, therefore $G$ is not a tree. Thus if the graph $G$ is a tree without infinite paths, the case (3) must occur, which was to be proved.

**Lemma 2b.** *If $G$ is a tree with infinite paths, then it can be directed so that at each vertex of $G$ there is at least one incoming edge of $G$, therefore $\emptyset \in \mathscr{M}(G)$.*

Proof. Choose an infinite path $P$ in $G$ and direct its edges so that it becomes a directed path. If $P$ is one-way infinite, let there be an incoming edge of $P$

at its end vertex. If $P$ is two-way infinite, its orientation may be arbitrary. If we remove all edges of $P$, we obtain a forest $G'$. Each component of $G'$ has exactly one common vertex with $P$. Direct each component of $G'$ so that this vertex might be the unique vertex at which there is no incoming edge (see Lemma 3). The orientation of $G$ thus obtained evidently satisfies the condition.

**Lemma 5a.** *Lemma 5 holds for all trees without infinite paths.*

Proof. At first let $Y_0$ be finite. Let $X_0$ be well-ordered of the type $\alpha$. Denote the vertices of $X_0$ by $u_\gamma$ for all $\gamma < \alpha$. For $\gamma \leqq \alpha$ let $X_0(\gamma) = \{u_\lambda/0 \leqq \lambda < \gamma\}$. Further let $G(\gamma)$ be the subgraph of $G$ consisting of all paths connecting two vertices of $X_0(\gamma) \cup Y_0$. Prove that all graphs $G(\gamma)$ can be directed so that $X_0(\gamma)$ (resp. $Y_0$) might be the set of all vertices at which there is no incoming (resp. outgoing) edge and if $\delta < \gamma$, then in this orientation all edges of $G(\delta)$ are directed in the same way in $G(\delta)$ as in $G(\gamma)$. For $\alpha = 1$ this holds according to Lemma 3 of [1]. Let $\gamma > 1$. Suppose that the affirmation holds for all $\lambda < \gamma$, where $\gamma$ is some ordinal number less or equal to $\alpha$. If $\gamma = \beta + 1$, where $\beta$ is some ordinal number, we take a path $P$ of maximal length in $G(\gamma)$ such that one of its end vertices is $u_\gamma$ and all its internal vertices (if any) have the degree 2. The end vertex of $P$ different from $u_\gamma$ denote by $w$. Evidently the degree of $w$ is at least three. Let $h_1$, $h_2$ be two different edges incident at $w$ and not belonging to $P$. Take two paths $P_1$ and $P_2$ connecting $w$ with some end vertices $x_1$ and $x_2$ of $G(\beta)$; the edge $h_1$ belongs to $P_1$, the edge $h_2$ belongs to $P_2$. Then the paths $P_1$ and $P_2$ cannot have any common vertex except $w$; in the reverse case a circuit would exist, which is impossible, as $G(\gamma)$ is a tree. The union of $P_1$ and $P_2$ is a path connecting two end vertices of $G(B)$ and containing $w$, hence according to the definition of $G(\beta)$ the vertex $w$ belongs to $G(\beta)$. Evidently other vertices of $P$ do not belong to $G(\beta)$. If $y$ is some vertex of $G(\gamma)$ not belonging to $G(\beta)$, it must be a vertex of some path $P_3$ connecting $u_\gamma$ with some vertex $x_3$ of $X_0(\beta) \cup Y_0$. Such a path $P_3$ evidently contains $w$. The path $P_3$ may contain at most one of the edges $h_1$, $h_2$; suppose without the loss of generality that it does not contain $h_1$. Then if we go from $x_1$ along $P_1$ to $w$ and then from $w$ along $P_3$ to $x_3$, we obtain a path $P_4$ from $x_1$ to $x_3$. If $y$ does not belong to $P$, it must lie on $P_3$ between $w$ and $x_3$ and therefore on $P_4$; according to the definition it belongs to $G(\beta)$. If $y$ belongs to $P$, it evidently does not belong to $G(\beta)$. So $G(\gamma)$ is the union of $G(\beta)$ and $P$. According to the induction assumption we can direct $G(B)$ according to our affirmation. Then we direct $P$ so that it might become a directed path from $u_\gamma$ to $w$. We have evidently obtained the desired orientation of $G(\gamma)$.

If $\gamma$ is a limit ordinal number, then $X_0(\gamma) = \bigcup_{\lambda<\gamma} X_0(\lambda)$ and evidently also

$G(\gamma)$ is the union of all $G(\lambda)$ for $\lambda < \gamma$. Therefore each edge of $G(\gamma)$ belongs to some $G(\lambda)$ for $\lambda < \gamma$ and we direct it in the same way as in $G(\lambda)$. According to the induction assumption this orientation does not depend on the choice of $\lambda$. If in this orientation there were a vertex not belonging to $X_0(\gamma)$ (resp. $Y_0$) at which there were no incoming (resp. outgoing) edge, this vertex would be contained in some $G(\lambda)$ for $\lambda < \gamma$ and in $G(\lambda)$ there would again be no incoming (resp. outgoing) edge at it and it would not be be contained in $X_0(\lambda)$ (resp. $Y_0$), contrary to the induction assumption. Hence the proof is finished for $Y_0$ finite. For $Y_0$ infinite we can proceed analogously by the transfinite induction according to the ordinal number of $Y_0$.

**Lemma 6a.** *Let $G$ be a tree with infinite paths. Given an arbitrary decomposition of the set of its end vertices into two disjoint subsets $X_0$, $Y_0$, the graph $G$ can be directed so that $X = X_0$, $Y = Y_0$.*

Proof. If $G$ contains a two-way infinite path $P$, let $G'$ be the subgraph of $G$ consisting of the path $P$ and all one-way infinite paths beginning in a vertex of $P$. $G'$ is a tree without end vertices. Direct it so that $P$ becomes a directed path and each component of the graph formed from $G$ by removing all edges of $P$ is directed so that its common vertex with $P$ is the unique vertex at which there is no incoming edge of that component.

Let $G''$ be a graph originating from $G$ by the removing of all edges of $G'$. If $G''$ is an empty graph, the proof is finished. If it is nonempty, each component $H$ of it has evidently no infinite paths. Let $X_0'$ (resp. $Y_0'$) be the set of vertices of $H$ which belong to $X_0$ (resp. to $Y_0$), let $u$ be the common vertex of $H$ and $G'$ (there is evidently only one). The set of end vertices of $H$ is $X_0' \cup Y_0' \cup \{u\}$. Decompose it into two disjoint subsets $X_0''$, $Y_0''$. If $X_0' \neq \emptyset$, then $X_0'' = X_0'$, $Y_0'' = Y_0' \cup \{u\}$. If $X_0' = \emptyset$, then $X_0'' = \{u\}$, $Y_0'' = Y_0'$. Then direct the graph $H$ so that $X = X_0''$, $Y = Y_0''$. Do it with each component of $G''$. An orientation satisfying the condition is obtained. If $G$ contains only one—way infinite paths, choose an infinite path $P'$ beginning in an end vertex of $G$ (such a path must exist). If $P'$ begins in a vertex $v$ of $X_0$ (resp. $Y_0$), direct it so that it becomes a directed path and at $v$ there is an outgoing (resp. incoming) edge of $P$. The components of the graph originating by the removing of all edges of $P$ are trees without infinite paths, hence we proceed as in the first case with $H$.

Now we can generalize Theorems 1 and 2 (as other considerations do not use the finiteness of $G$).

**Theorem 1a—2a.** *Theorems* 1 *and* 2 *of* [1] *hold also for infinite graphs if we substitute the expression „tree" by the expression „tree with a finite diameter".*

REFERENCE

[1] Zelinka B., *Introducing an orientation into a given non-directed graph*, Mat.-fyz. časop. *16* (1966), 66—71.

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