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CENTER OF A BOUNDED LATTICE

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Let L be a complete lattice and let $C(L)$ be the center of L. Several authors have found sufficient conditions for $C(L)$ to be a closed sublattice of L. Each of the following conditions (a) — (d) is sufficient for $C(L)$ to be closed:

(a) L is a continuous geometry (von Neumann [10]).

(b) L is orthocomplemented modular (Kaplansky [8]).

(c) L is orthomodular (Foulis [2] Holland [3]).

(d) L is relatively semi-orthocomplemented (Maeda [9]).

All the above results were generalized by Janowitz [7] who proved that a sufficient condition is

(e) L is relatively complemented.

In this note it will be shown that a sufficient condition for the closedness of $C(L)$ can be expressed by using properties of congruence relations on L; namely the following condition is sufficient:

(f) Any two congruence relations on L are permutable and if $a, b\ c, d \in L$, $a \leqq b < c \leqq d$, $b\, R\, c$ for some congruence relation R on L, then there are elements $b_1, c_1 \in L$ such that $a < b_1 \leqq d$, $a \leqq c_1 < d$, $a\, R\, b_1$, $c_1\, R\, d$.

It is easy to verify that (e) implies (f) and that (e), (f) are not equivalent (cf. Lemma 9); hence our result is a generalization of that of Janowitz.

Janowitz (loc. cit.) remarks that it is not known if the centre of a complete lattice must be a closed sublattice (cf. also Birkhoff [1], Problem 34).

In [5] it was proved that the centre of each infinitely distributive complete lattice is a closed sublattice and that there exists a nondistributive complete lattice L such that $C(L)$ is not closed in L. In [6] it was shown that there exists a distributive complete lattice whose centre fails to be closed and that the following condition is necessary and sufficient for $C(L)$ to be a closed sublattice of L:

(g) If $x, y \in L$, $x \geqq y$, $\{a_\alpha\} \subset C(L)$, then

(1) $y \vee (x \wedge (\wedge\, a_\alpha)) = \wedge\, (y \vee (x \wedge a_\alpha))$,

(2) $x \wedge (y \vee (\vee\, a_\alpha)) = \vee\, (x \wedge (y \vee a_\alpha))$.

In Thm. 3 of [6] this result was applied for the investigation of direct factors of a coditionally complete lattice.

For the basic notions and denotations cf. Birkhoff [1].

Let L be a bounded lattice that need not be complete, $L = [u, v]$. We denote by $\Theta(L)$ the set of all congruence relations on L. The least and the greatest element of $\Theta(L)$ will be denoted by $\bar{0}$ and $\bar{1}$, respectively. For any $x \in L$ and any $R \in \Theta(L)$ we denote by $x(R)$ the set of all $y \in L$ with $y\,R\,x$.

Let I be a nonempty set and for each $i \in I$ let R_i and R'_i be congruence relations on L such that

(i) R_i and R'_i are permutable,

(ii) $R_i \wedge R'_i = \bar{0}$,

(iii) $R_i \vee R'_i = \bar{1}$.

From (i), (ii) and (iii) we get (cf. [1], p. 164, Thm. 5) that the correspondence

$$x \to (x(R_i),\quad x(R'_i)) \qquad (x \in L) \tag{3}$$

is an isomorphism of L onto the direct product $A \times B$, where $A = L/R_i$, $B = L / R'_i$.

From (i) and (iii) it follows that there exist elements c_i, $c'_i \in [u, v]$ such that

$$uR_ic_i, \quad c_iR'_iv,$$

$$uR'_ic'_i, \quad c'_iR_iv.$$

Moreover, from (ii) we obtain that the elements c_i and c'_i are uniquely determined and that

$$c_i \wedge c'_i = u, \quad c_i \vee c'_i = v.$$

Assume that the elements

$$\wedge\, c_i = c, \quad \vee\, c'_i = d$$

do exist in L. Hence c, $d \in [u, v]$. Denote $R = \wedge\, R_i$ and let R' be the least congruence on L in which c and v belong to the same class.

Lemma 1. *Let $i \in I$. If $x \in [u, v]$, $x \in u(R_i)$, then $x \leqq c_i$.*

Proof. From $x \in u(R_i)$ we obtain $x(R_i) = u(R_i)$. Because of $x \leqq v$ we have $x(R'_i) \leqq v(R'_i) = c_i(R'_i)$. Therefore, since (3) is an isomorphism and $c_i(R_i) = x(R_i)$, we infer that $x \leqq c_i$.

The assertion dual to Lemma 1 can be proved analogously.

Lemma 2. *c is the greatest element of the set $u(R)$.*

Proof. We have $u \leqq c \leqq c_i$ for each $i \in I$, thus uR_ic and hence $c \in u(R)$. Let $x \in u(R)$. Then $x \in u(R_i)$ and so by Lemma 1, $x \leqq c_i$ for each c_i; therefore $x \leqq c$.

In a dual way we obtain

Lemma 2'. *d is the least element of the set $v(R)$.*

Lemma 3. *Assume that L fulfils (f). Then $c \wedge d = u$, $c \vee d = v$.*

Proof. Put $c \wedge d = \bar{u}$ and suppose that $u < \bar{u}$. Then $uR\bar{u}$ and hence it follows from (f) that there exists $u_1 \in L$ with $u \leqq u_1 < d$, u_1Rd, but this contradicts Lemma 2′. Therefore $c \wedge d = u$. In a dual way we can verify that $c \vee d = v$.

Lemma 4. *Assume that L fulfils (f). Let $[p, x]$, $[y, q]$ be transposed intervals of L, $[p_1, x_1] \subset [p, x]$, $p_1 < x_1$. Let S be a congruence relation on L such that p_1Sx_1. Then there is $t \in L$ with $y < t \leqq q$ such that ySt.*

Proof. We may suppose that $x \wedge y = p$, $x \vee y = q$ (in the case $p \wedge q = y$, $p \vee q = x$ we proceed dually). From (f) it follows that there is $z \in L$ such that $p < z \leqq x$, pSz. Denote $z \vee y = t$. Then $y < t \leqq q$, ySt.

Analogously we can verify the dual assertion.

Let I and I' be intervals of L. The interval I is said to be weakly projective to I' if there are intervals $I_0, I_1, \ldots, I_n$ of L such that $I_0 = I$, $I_n = I'$ and I_{j+1} is a subinterval of an interval that is transposed to I_j $(j = 0, 1, \ldots, n - 1)$.

Lemma 5. *Let $x, y \in L$, $x < y$. Then $x\ R'\ y$ if and only if there are elements $x_0, x_1, \ldots\ x_n \in L$ such that $x_0 = x$, $x_n = y$, $x_j < x_{j+1}$ and $[c, v]$ weakly projective to $[x_j, x_{j+1}]$ for $j = 0, 1, \ldots, n - 1$.*

This follows from [4], § 1.

By induction we obtain from Lemma 4 and Lemma 5:

Lemma 6. *Assume that L fulfils (f). Let $x, y \in L$, $x < y$, $x\ R'\ y$. Let R_2 be a congruence relation on L such that xR_2y. Then there is $t \in L$ such that $c < t \leqq \leqq v$, cR_2t.*

Lemma 7. *If L fulfils (f), then $R \wedge R' = \bar{0}$.*

Proof. Suppose that $R \wedge R' \neq \bar{0}$. Then there are elements $x, y \in L$ with $x < y$, $x(R \wedge R')\, y$. Put $R \wedge R' = R_2$ and let t be as in Lemma 6. We have $c < t \leqq v$ and cRt, which is a contradiction (cf. Lemma 2).

We have $uRcR'v$, hence $u(R \vee R')v$ and therefore

$$R \vee R' = \bar{1}\,. \tag{4}$$

Lemma 8. *Assume that L fulfils (f). The correspondence*

$$x \to (x(R'),\quad x(R)) \qquad (x \in L)$$

is an isomorphism of L onto $(L/R') \times (L/R)$.

This follows from (4), and Lemma 7 from the fact that any two congruence relations on L are permutable.

The notion of the center $C([u, v])$ of the lattice $L = [u, v]$ is defined as follows. An element $c \in L$ belongs to $C([u, v])$ if and only if there are lattices A, B and an isomorphism φ of L onto the direct product $A \times B$ such that, if we denote $\varphi(z) = (z_A, z_B)$ for each $z \in L$, then

$$c_A = v_A, \quad c_3 = u_B .$$

Each element $c \in C([u, v])$ has exactly one relative complement in the interval $[u, v]$; this relative complement will be denoted c'.

Another way of defining the set $C([u, v])$ is as follows (it is a direct consequence of [1], p. 164, Thm. 5): An element $c_i \in L$ belongs to $C([u, v])$ if and only if there are $R(c_i)$, $R'(c_i) \in \Theta(L)$ such that $R(c_i)$, $R'(c_i)$ are permutable, $R(c_i) \wedge R'(c_i) = 0$, $R(c_i) \vee R'(c_i) = \bar{1}$ and $uR(c_i)c_i$, $c_iR'(c_i)v$.

Theorem 1. *Let $L = [u, v]$ be a lattice fulfilling (f), $\emptyset \neq \{c_i\} \subset C([u, v])$. Assume hat the elements $\wedge c_i$, $\vee c_i'$ exist in L. Then $\wedge c_i$, $\vee c_i' \in C$ $([u, v])$.*

Proof. Denote $\wedge c_i = c$, $\vee c_i' = d$. Let $R(c_i)$ and $R'(c_i)$ be as in the definition of the center $C([u, v])$. Denote $R(c_i) = R_i$, $R'(c_i) = R_i'$. By means of the congruence relations R_i, R_i' we construct the congruence relations R and R' as above. Then we have

$$u\,R\,c, \quad c\,R'\,v, \quad u\,R'\,d, \quad d\,R\,v .$$

From this and from (4) and Lemmas 7, 8 we infer that c and d belong to $C([u, v])$.

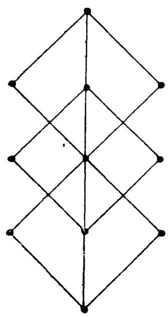

Fig. 1

Corollary 1. *Let L be a complete lattice fulfilling (f). Then $C(L)$ is a closed sublatice of L.*

Lemma 9. *For any lattice L, (e) implies (f). The conditions (e), (f) are not equivalent.*

Proof. Let L be a relatively complemented lattice. It is well known that any two congruence relations on L are permutable ([1], p. 163). Let $a, b, c, d \in L$, $a \leqq b < c \leqq d$, bRc for some congruence relation R on L. Let b_1 be a relative complement of b in the interval $[a, c]$. Then $a < b_1 \leqq c$ and aRb_1. In a dual way we can find $c_1 \in L$ such that $a \leqq c_1 < d$, c_1Rd. Hence L fulfils (f).

Let L be the lattice in Fig. 1. The lattice L is modular and any two prime intervals of L are projective. Thus L has no nontrivial congruence relations and hence L fulfils (f). On the other hand, L is not relatively complemented.

From Lemma 9 and Corollary 1 we obtain:

Corollary 2. (Janowitz [7]) *Let L be a complete relatively complemented lattice. Then $C(L)$ is a closed sublattice of L.*

Let us remark that the condition (g) does not appear to be an immediate consequence of the condition (e). We can use (g) for getting the well-known result on the infinite distributivity of a complete Boolean algebra. In fact, if L is a complete Boolean algebra, then $C(L) = L$ and hence by Thm. 2, [6], the condition (g) is valid. By putting $x = 1$ in (1) and $y = 0$ in (2) we get the infinite distributive laws for L.

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