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1. Introduction

Let G mean the open Grassmannian of all lines (not only those passing through the origin O) in the $(n + 1)$-dimensional Euclidean space R^{n+1}. Let Kb^n be a subspace of G which consists of all lines x parallel to a given Euclidean plane $\mathrm{R}^2 \subset R^{n+1}$, $n \geqq 2$ and tangent to the unit sphere $S^n \subset R^{n+1}$. Since Kb^2 is homeomorphic to the Klein bottle (see [2]) there is good reason for a

Definition. *The topological subspace* Kb^n *of* G *is called the Klein hyperbottle in* R^{n+1}.

The aim of this paper is:

(i) to find a CW-decomposition of Kb^n (see Theorem 1) and

(ii) to calculate the homology groups $\mathrm{H}_i(\mathrm{Kb}^n)$ (see Theorem 2).

2. Notation

$S^n = \{x \in \mathrm{R}^{n+1} \mid |x| = 1\}$ is the unit n-sphere;

S^1 the unit circle is identified with $\{e^{it} \mid t \in \mathrm{R}\} \subset \mathrm{C}$;

$\mathrm{B}^n = \{x \in \mathrm{R}^n \mid |x| \leqq 1\}$ is the closed n-dimensional unit ball with the boundary $\partial\mathrm{B}^n = \mathrm{S}^{n-1}$;

$\langle a, b\rangle$, $a, b \in \mathrm{R}^{n+1}$, $b \neq 0$ means a line $\{a + \lambda b \mid \lambda \in \mathrm{R}\} \in \mathrm{G}$;

$\alpha\colon \mathrm{C} \to \mathrm{R}^{n+1}$, $x + \mathrm{i}y \to (x, y, 0, \ldots, 0)$ and

$\beta\colon \mathrm{R}^{n-1} \to \mathrm{R}^{n+1}$, $(x^1, \ldots, x^{n-1}) \to (0, 0, x^1, \ldots, x^{n-1})$ inclusion maps;

$\gamma\colon (\mathrm{B}^{n-2}, \partial\mathrm{B}^{n-2}) \to (\mathrm{S}^{n-2}, -k)$, $w \to 2\sqrt{1 - |w|^2}w + (2|w|^2 - 1)k$,

where $k = (0, \ldots, 0, 1) \in \mathrm{R}^{n-1}$;

* is the distinguished point of a factor-space X/Y

3. The CW-decomposition of Kb^n

The Klein hyperbottle Kb^n will be expressed in a more suitable form, namely as a factor-space of $\mathrm{S}^1 \times \mathrm{B}^{n-1}$. Then a CW-decomposition of Kb^n is described.

Lemma 1. *Let Θ be an equivalence relation on* $S^1 \times B^{n-1}$ *given as follows:*

$$(u, v)\Theta(u', v') \Leftrightarrow (u = u' \text{ and } v = v') \text{ or } (u + u' = 0 \text{ and } v = v' \in \partial B^{n-1}).$$

Then the factor-space $K = (S^1 \times B^{n-1})/E$ *is homeomorphic to the Klein hyper-bottle* Kb^n.

Proof. It is not difficult to show that the map

$$\sigma : S^1 \times B^{n-1} \to Kb^n,\ (u, v) \to \langle \sqrt{1 - |v|^2}\alpha(u) + \beta(v), \alpha(iu)\rangle$$

is both, well-defined and surjective. Moreover, $\sigma(u, v) = \sigma(u', v')$ holds if and only if

$$\sqrt{1 - |v|^2}\alpha(u) + \beta(v) = \sqrt{1 - |v'|^2}\alpha(u') + \beta(v') \text{ and } iu = \pm iu',$$

i.e. $\qquad v = v'$ and $(u = u'$ or $u = -u',\ v \in \partial B^{n-1})$.

Hence Kb^n, as a σ-image of $S^1 \times B^{n-1}$ is homeomorphic to the factor-space $K = S^1 \times B^{n-1}/\Theta$.

Theorem 1. *The Klein hyperbottle* $Kb^n \approx K = S^1 \times B^{n-1}/\Theta$ *admits a CW-decomposition into six disjoint cells of the dimension* $0, 1, n-2, n-1, n-1$ *and* n. *Characteristic maps are*

$e^0 \equiv f^0\colon B^0 \to K,\ 0 \to [\pm 1, v_0]$, *where* $v_0 = (1, 0, \dots, 0) \in R^n$
$e^1 \equiv f^1\colon B^1 \to K,\ t \to [\pm e^{\frac{\pi}{2}i(1-t)}, v_0]$,
$e^{n-2} \equiv f^{n-2}\colon B^{n-2} \to K,\ w \to [\pm 1, \gamma(w)]$,
$e_1^{n-1} \equiv f_1^{n-1}\colon B^{n-1} \to K,\ v \to [-1, v]$,
$e_2^{n-1} \equiv f_2^{n-1}\colon B^1 \times B^{n-2} \to K,\ (t, w) \to [e^{\frac{\pi}{2}i(1-t)}, \gamma(w)]$,
$e^n \equiv f^n\colon B^1 \times B^{n-1} \to K,\ (t, v) \to [e^{-\pi i t}, v]$,
where $e^i \equiv f^i\colon B^i \to K$ *means* $e^i = f^i\ (\text{int } B^i)$.

Corollary. *The Euler—Poincare characteristic of* Kb^n *is zero.*

Proof. See [1] Proposition 5.9 p. 105.

4. Groups $H_i(Kb^n)$

We compute the cellular boundary in the Klein hyperbottle $Kb^n = K$. First of all it is obvious that

$$(1) \qquad \partial e^0 = 0, \quad \partial e^1 = 0, \quad \partial e^{n-2} = 0 \quad \text{and} \quad \partial e_1^{n-1} = \pm e^{n-2}.$$

Thus only the three incidence numbers, namely $[e_2^{n-1} : e^{n-2}]$, $[e^n : e_1^{n-1}]$ and $[e^n : e_2^{n-1}]$ are in need of being computed.

Lemma 2. $[e_2^{n-1} : e^{n-2}] = 0$, *therefore* $\partial e_2^{n-1} = 0$.

Proof. The incidence number $[e_2^{n-1} : e^{n-2}]$ is defined as the degree of the map

$$\Phi : S^{n-2} = \partial(B^1 \times B^{n-2}) \xrightarrow{\varphi} K^{n-2} \xrightarrow{p} K^{n-2}/(K^{n-2} - e^{n-2}) = S^{n-2}$$

which is the composition of the attaching map φ for e_2^{n-1} and the canonical projection p of the $(n - 2)$-skeleton K^{n-2} of K. To check the number deg Φ let us consider a map

$$h : S^{n-2} = \partial(B^1 \times B^{n-2}) \to S^{n-2} = \partial(B^1 \times B^{n-2}),\ (t, w) \to (-t, w)$$

regarded as the involution on S^{n-2}. Since $\deg h = -1$, and $\Phi = \Phi \circ h$, it is $\deg \Phi = \deg \Phi . \deg h = -\deg \Phi$, hence $\deg \Phi = 0$.

Lemma 3. $[e^n : e_1^{n-1}] = 0$ and $[e^n : e_2^{n-1}] = \pm 2$, *therefore is* $\partial e^n = \pm 2e_2^{n-1}$.

Proof. The first of these two assertions may be proved by the same argument as that of Lemma 2. The second assertion follows immediately from the characteristic maps f^n and f_2^{n-1}.

Theorem 2. *The homology groups of the Klein hyperbottle* Kb^n *are*

$$n = 2:\ H_0(Kb^2) = Z,\ H_1(Kb^2) = Z + Z_2,\ H_i(Kb^2) = 0 \text{ for } i \neq 0,1$$

$$n > 3: \begin{cases} H_0(Kb^n) = Z.\ H_1(Kb^n) = Z,\ H_{n-1}(Kb^n) = Z_2, \\ H_i(Kb^n) = 0, \text{ for } i \neq 0,1,\ n - 1. \end{cases}$$

REFERENCES

[1] DOLD, A.: Lectures on Algebraic Topology. Springer-Verlag 1972.

[2] HEJNÝ, M.: Models of the Klein Bottle. Acta Fac. Rerum. nat. Univ. Com. Math. (to appear).

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