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ON THE GENERAL PROBLEM OF ADJUSTMENT OF MEASURED VALUES

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The aim of the present paper is the determination of the estimation ([1], [3]) of components of the N dimensional vector $\mathbf{y}$ and the k-dimensional vector $\mathbf{z}$ which satisfy q conditions

$$\mathbf{x}_0 + \mathbf{X}\mathbf{y} + \mathbf{X}_1\mathbf{z} = \mathbf{0}\,. \tag{1}$$

It is assumed that the vector $\mathbf{x}_0$ and the matrices $\mathbf{X}$ and $\mathbf{X}_1$ are known. The rank of the matrix $(\mathbf{X}, \mathbf{X}_1)$ is $h(\mathbf{X}, \mathbf{X}_1) = q$ and for the matrix $\mathbf{X}_1$ we have $h(\mathbf{X}_1) = k < q$, analogously. The condition $N > q - k > 0$ holds for N*).

The components of the vector $\mathbf{y}$ can be measured and results of the measurement $l_1, \dots, l_N$ of the components $y_1, \dots, y_N$ are the components of realization of the random vector $\mathbf{l} \dots N(\mathbf{y}, \sigma^2\mathbf{P}^{-1})$. A diagonal matrix $\mathbf{P}$ is the matrix of weights $p_i > 0$ of the results l_i, $i = 1, \dots, N$. The components z_j, $j = 1, \dots, k$ of the vector $\mathbf{z}$ cannot be measured.

The above problem is sometimes called the adjustment of conditions with parameters ([2], [4]). The next problem is to determine how the statistical properties of calculated estimations are or how connections among them are.

Lemma 1. *Let* $\mathbf{A}$ *be a matrix of a positive definite quadratic form of the order* $N \times N$ *and let* $\mathbf{R}$ *be a matrix of the order* $q \times N$ *with the rank* $s \leqslant \min(q, N)$. *Then* $\mathbf{RAR}'$ *is of the rank* s.

(Proof in [1], p. 41.)

Corollary. *The matrix* $\mathbf{XP}^{-1}\mathbf{X}'$ *is of the rank* $h(\mathbf{X})$.

Lemma 2. *The matrix*

* If $N = q - k$, the system (1) would be a system of q equations with q unknows. According to the assumption that the matrix of system (1) is regular, the problem of the determination of the maximum likelihood estimation would be trivial. Therefore this possibility, will not be dealt with.

(2) $$\begin{pmatrix} \mathbf{XP}^{-1}\mathbf{X}' & \mathbf{X}_1 \\ \mathbf{X}_1' & \mathbf{O} \end{pmatrix}$$

is of the rank $q + k$ and is therefore regular.

Proof. $h(\mathbf{X}_1) = k$, $h(\mathbf{X}, \mathbf{X}_1) = q$ hence $h(\mathbf{X}) \geqslant q - k$. According to Lemma 1, $h(\mathbf{XP}^{-1}\mathbf{X}') \geqslant q - k$ therefore $h(\mathbf{XP}^{-1}\mathbf{X}', \mathbf{X}_1) \geqslant q$, since linearly independent columns of the first submatrix remain linearly independent on the columns of the other submatrix. According to the first size of submatrices, $h(\mathbf{XP}^{-1}\mathbf{X}', \mathbf{X}_1) = q$ holds. Analogously in a matrix

$$\begin{pmatrix} \mathbf{XP}^{-1}\mathbf{X}' \\ \mathbf{X}_1 \end{pmatrix}$$

k rows of the submatrix $\mathbf{X}_1'$ are linear independent on linearly independent rows of the submatrix $\mathbf{XP}^{-1}\mathbf{X}'$ hence all rows of the whole matrix (2) are linearly independent.

Theorem 1. *The conditional local extreme of a likelihood function holds for* $\mathbf{y} = \tilde{\boldsymbol{l}}$ *and* $\mathbf{z} = \tilde{\mathbf{z}}$, *resp., which satisfy equations*

$$\left(\begin{array}{c|c} \mathbf{XP}^{-1}\mathbf{X}' & \mathbf{X}_1 \\ \hline \mathbf{X}_1' & \mathbf{O} \end{array}\right) \begin{pmatrix} \boldsymbol{k} \\ \tilde{\mathbf{z}} \end{pmatrix} + \begin{pmatrix} \boldsymbol{m} \\ \mathbf{O} \end{pmatrix} = \mathbf{O}\,,$$

$$\begin{aligned} \boldsymbol{m} &= \boldsymbol{x}_0 + \mathbf{X}\boldsymbol{l}\,, \\ \tilde{\mathbf{v}} &= \mathbf{P}^{-1}\mathbf{X}'\boldsymbol{k}\,, \\ \tilde{\boldsymbol{l}} &= \boldsymbol{l} + \tilde{\mathbf{v}}\,. \end{aligned}$$

Proof. The likelihood function gains its local extreme when the corrections v_i, $i = 1, \ldots, N$, i. e. components of the vector $\mathbf{v}$ and values Z_j, $j = 1, \ldots, k$ i. e. components of the vector $\mathbf{z}$, provide the minimum of the function.

$$\Phi(v_1, \ldots, v_N; z_1, \ldots, z_k) = \mathbf{v}'\mathbf{P}\mathbf{v} - 2\boldsymbol{k}'(\mathbf{X}\mathbf{v} + \mathbf{X}_1\mathbf{z} + \boldsymbol{m})\,,$$

where $\boldsymbol{k}$ is the q-dimensional vector of the Lagrange coefficients.

$$\mathrm{d}\Phi = 2(\mathbf{v}'\mathbf{P}, \mathbf{O}')\,\mathrm{d}\left(\frac{\mathbf{v}}{\mathbf{z}}\right) - 2\boldsymbol{k}'(\mathbf{X}, \mathbf{X}_1)\,\mathrm{d}\left(\frac{\mathbf{v}}{\mathbf{z}}\right) = \mathbf{O}_{N+k,1}\,,$$

which shows that $(\tilde{\mathbf{v}}'\mathbf{P}, \mathbf{O}') = \boldsymbol{k}'(\mathbf{X}, \mathbf{X}_1)$ and therefore $\tilde{\mathbf{v}} = \mathbf{P}^{-1}\mathbf{X}'\boldsymbol{k}$; $\mathbf{X}_1'\boldsymbol{k} = \mathbf{O}$. If the two last relationships are considered in the system of conditions (1), then

$$\begin{aligned} \boldsymbol{x}_0 + \mathbf{X}(\boldsymbol{l} + \mathbf{P}^{-1}\mathbf{X}'\boldsymbol{k}) + \mathbf{X}_1\mathbf{z} &= \mathbf{O}_{q,1} \\ \mathbf{X}_1'\boldsymbol{k} &= \mathbf{O}_{k1} \end{aligned}$$

holds, which completes the proof.

Theorem 2. *The extreme from Theorem 2 is the minimum of the function Φ.*

Proof. Let

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{11}, & \mathbf{Q}_{12} \\ \mathbf{Q}_{21}, & \mathbf{Q}_{22} \end{pmatrix}. \tag{3}$$

be a reciprocal matrix of (2).

Then

$$\begin{aligned} \mathbf{G}\mathbf{Q}_{11} + \mathbf{X}_1\mathbf{Q}_{21} &= \mathbf{I}_{qq}\,, \\ \mathbf{G}\mathbf{Q}_{12} + \mathbf{X}_1\mathbf{Q}_{22} &= \mathbf{O}_{qk}\,, \\ \mathbf{X}_1'\mathbf{Q}_{11} &= \mathbf{O}_{kq}\,, \\ \mathbf{X}_1'\mathbf{Q}_{12} &= \mathbf{I}_{kk} \end{aligned} \tag{4}$$

holds, where $\mathbf{G} = \mathbf{X}\mathbf{P}^{-1}\mathbf{X}'$.

Further the following holds:

$$\tilde{\mathbf{v}}'\mathbf{P}\tilde{\mathbf{v}} = \boldsymbol{k}'\mathbf{G}\boldsymbol{k} = \boldsymbol{m}'\mathbf{Q}_{11}\mathbf{G}\mathbf{Q}_{11}\boldsymbol{m} = (\mathbf{X}\tilde{\mathbf{v}} + \mathbf{X}_1\tilde{\mathbf{z}})'\mathbf{Q}_{11}(\mathbf{X}\tilde{\mathbf{v}} + \mathbf{X}_1\tilde{\mathbf{z}}) = \tilde{\mathbf{v}}'\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\tilde{\mathbf{v}}.$$

Theorem 1 and the first and third equality in (4) were applied.

Now it will be shown that for another choice of vectors $\mathbf{v}$ and $\mathbf{z}$, which obviously have to satisfy the conditions (1), $\mathbf{v}'\mathbf{P}\mathbf{v} \geqslant \mathbf{v}'\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{v} = \boldsymbol{m}'\mathbf{Q}_{11}\boldsymbol{m} = \tilde{\mathbf{v}}'\mathbf{P}\tilde{\mathbf{v}}$ will hold.

The following holds:

$$\mathbf{v}'\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{v} = \mathbf{v}'\mathbf{P}^{\frac{1}{2}}\mathbf{P}^{-\frac{1}{2}}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-\frac{1}{2}}\mathbf{P}^{\frac{1}{2}}\mathbf{v}\,; \quad \mathbf{P}^{\frac{1}{2}}\mathbf{v} = \mathrm{w}\,;$$

$\mathbf{P}^{-\frac{1}{2}}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-\frac{1}{2}} = \mathbf{U}$; $\mathbf{U}' = \mathbf{U}$; $\mathbf{U}^2 = \mathbf{U}$. There is such an orthogonal matrix $\mathbf{F}$, for which the following holds: $\mathbf{F}\mathbf{U}\mathbf{F}' = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix and $\mathbf{D}^2 = \mathbf{D}$, which means that the diagonal elements of the matrix $\mathbf{D}$ are only zeros or unities. If $\mathbf{F}\mathbf{w}$ is denoted by $\mathbf{t}$, the following holds: $\mathbf{v}'\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{v} = \mathbf{w}'\mathbf{F}'\mathbf{F}\mathbf{U}\mathbf{F}'\mathbf{F}\mathbf{w} = \mathbf{t}'\mathbf{D}\mathbf{t}$. For the rank of the matrix $\mathbf{D}$ we have: $h(\mathbf{D}) = \mathrm{Sp}(\mathbf{D}) = \mathrm{Sp}(\mathbf{F}\mathbf{P}^{-\frac{1}{2}}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-\frac{1}{2}}\mathbf{F}') = \mathrm{Sp}(\mathbf{X}\mathbf{P}^{-\frac{1}{2}}\mathbf{F}'\mathbf{F}\mathbf{P}^{-\frac{1}{2}}\mathbf{X}'\mathbf{Q}_{11}) = \mathrm{Sp}(\mathbf{I}_{qq} - \mathbf{X}_1\mathbf{Q}_{21}) = q - \mathrm{Sp}(\mathbf{Q}_{21}\mathbf{X}_1) = q - \mathrm{Sp}(\mathbf{I}_{kk}) = q - k$.

The relationships (4) and the rule $\mathrm{Sp}(\mathbf{A}_{pq}\mathbf{B}_{qp}) = \mathrm{Sp}(\mathbf{B}_{qp}\mathbf{A}_{pq})$ were applied. For $\mathbf{v}'\mathbf{P}\mathbf{v}$ we have: $\mathbf{v}'\mathbf{P}\mathbf{v} = \mathbf{v}'\mathbf{P}^{\frac{1}{2}}\mathbf{F}'\mathbf{F}\mathbf{P}^{\frac{1}{2}}\mathbf{v} = \mathbf{t}'\mathbf{t}$ and always $\mathbf{v}'\mathbf{P}\mathbf{v} = \mathbf{t}'\mathbf{t} \geqslant \mathbf{t}'\mathbf{D}\mathbf{t} = \mathbf{v}'\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{v} = \boldsymbol{m}'\mathbf{Q}_{11}\boldsymbol{m} = \tilde{\mathbf{v}}'\mathbf{P}\tilde{\mathbf{v}}$.

Theorem 3. *The vector $\boldsymbol{k}$ is a normal vector with at least q-k independent components $\boldsymbol{k} \ldots N(0, \sigma^2\mathbf{Q}_{11})$.*

Proof. The vector $\Delta = \boldsymbol{l} - \mathbf{y}$ will be called an error vector. Obviously $\Delta \ldots N(0, \sigma^2\mathbf{P}^{-1})$. For $\boldsymbol{k}$ we have: $\boldsymbol{k} = -\mathbf{Q}_{11}(\mathbf{x}_0 + \mathbf{X}\mathbf{y} + \mathbf{X}\Delta) = -\mathbf{Q}_{11}(-\mathbf{X}_1\mathbf{z} + \mathbf{X}\Delta) = -\mathbf{Q}_{11}\mathbf{X}\Delta$ (with regard to $\mathbf{Q}_{11}\mathbf{X}_1 = 0_{qk}$). The mean value $M(\boldsymbol{k}) = -\mathbf{Q}_{11}\mathbf{X}M(\Delta) = \mathbf{O}$ and for the covariance matrix of the vector $\boldsymbol{k}$ the following holds: $\Sigma_k = \mathbf{Q}_{11}\mathbf{X}.\ M(\Delta\Delta')\,\mathbf{X}'\mathbf{Q}_{11} = \sigma^2\mathbf{Q}_{11}\mathbf{G}\mathbf{Q}_{11}$. With regard to Lemma 1 we have for the rank of the matrix $\mathbf{Q}_{11}$:

$h(\mathbf{Q}_{11}\mathbf{X}) = h(\mathbf{Q}_{11})$. Further, with regard to (4), $\mathbf{Q}_{21}\mathbf{X}_1\mathbf{Q}_{21} = \mathbf{Q}_{21}$ and $\mathbf{X}_1\mathbf{Q}_{21}\mathbf{X}_1 = \mathbf{X}_1$, which shows $h(\mathbf{Q}_{21}) = h(\mathbf{X}_1) = k$. As the matrix $\mathbf{Q}$ is regular, $h(\mathbf{Q}_{11}, \mathbf{Q}_{12}) = q$ and therefore $h(\mathbf{Q}_{11}) \geqslant q - k$.

Theorem 4. *The vector $\tilde{\mathbf{z}}$ is a normal vector with at most k independent components $\tilde{\mathbf{z}} \dots N(\mathbf{z}; \sigma^2(-\mathbf{Q}_{22}))$.*

Proof. $\tilde{\mathbf{z}} = -\mathbf{Q}_{21}(\mathbf{x}_0 + \mathbf{X}\mathbf{y} + \mathbf{X}\Delta) = \mathbf{Q}_{21}\mathbf{X}_1\mathbf{z} - \mathbf{Q}_{21}\mathbf{X}\Delta$. With regard to the last equation of (4) the following holds

$$\tilde{\mathbf{z}} - \mathbf{z} = -\mathbf{Q}_{21}\mathbf{X}\Delta;\ \Sigma_{\tilde{\mathbf{z}}} = \mathbf{Q}_{21}\mathbf{X}\sigma^2\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{12} = \sigma^2\mathbf{Q}_{21}\mathbf{G}\mathbf{Q}_{12} = -\sigma^2\mathbf{Q}_{22},$$

with regard to the second equation of (4). With regard to Lemma 1 $h(\mathbf{Q}_{21}\mathbf{X}) = h(\mathbf{Q}_{22}) \leqslant k$.

Lemma 3. *If $\mathbf{s} = \mathbf{A}\mathbf{x}$ and $\mathbf{t} = \mathbf{B}\mathbf{x}$, where $\mathbf{x} \dots N(\mu, \Sigma_{\mathbf{x}})$, then $\mathbf{A}\,\Sigma_{\mathbf{x}}\mathbf{B}' = \mathbf{O}$ is the sufficient and necessary condition for the statistical independence of the vectors $\mathbf{s}$ and $\mathbf{t}$.* (proof see in [3] p. 57.)

Theorem 5. *The vector $\tilde{\mathbf{l}} = \mathbf{l} + \tilde{\mathbf{v}}$ and the vector $\tilde{\mathbf{v}}$ are statistically independent.*

Proof. $\tilde{\mathbf{l}} - \mathbf{y} = (\mathbf{I} - \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X})\Delta$; $\tilde{\mathbf{v}} = -\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\Delta$. Next we have $(\mathbf{I} - \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X})\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1} = \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{G}\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1} = \mathbf{O}_{NN}$, with respect to the equation $\mathbf{Q}_{11}\mathbf{G}\mathbf{Q}_{11} = \mathbf{Q}_{11}$ which proves this theorem with regard to Lemma 3.

Theorem 6. *The vector $\mathbf{z}$ and the vector $\tilde{\mathbf{v}}$ are statistically independent.*

Proof. $\tilde{\mathbf{z}} - \mathbf{z} = -\mathbf{Q}_{21}\mathbf{X}\Delta$; $\tilde{\mathbf{v}} = -\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\Delta$. With regard to the relationships $\mathbf{Q}_{21}\mathbf{G} = -\mathbf{Q}_{22}\mathbf{X}_1'$ and $\mathbf{X}_1'\mathbf{Q}_{11} = \mathbf{O}$, the following holds $\mathbf{Q}_{21}\mathbf{X}\sigma^2 \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1} = \mathbf{O}$, which proves this theorem with regard to Lemma 3.

Theorem 7. *Vector $\tilde{\mathbf{l}}$ is a singular normal vector with $N - (q - k)$ independent components $\tilde{\mathbf{l}} \dots N(\mathbf{y}; \sigma^2(\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1}))$.*

Proof. $\tilde{\mathbf{l}} - \mathbf{y} = (\mathbf{I} - \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X})\Delta$, therefore $\Sigma_{\tilde{\mathbf{l}}} = \sigma^2(\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1})$. The rank of the matrix $\mathbf{I} - \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}$ is denoted by h. We have $h = h(\mathbf{I} - \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}) = h[\mathbf{P}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X})\mathbf{P}^{-\frac{1}{2}}] = h(\mathbf{I} - \mathbf{U})$, where $\mathbf{U}$ is the matrix from the proof of Theorem 2. There is such an ortogonal matrix $\mathbf{F}$ that $\mathbf{F}\mathbf{U}\mathbf{F}'$ is a diagonal matrix $\mathbf{D}$, which satisfies the condition $\mathbf{D}^2 = \mathbf{D}$. Since we can easily obtain $h = h(\mathbf{I} - \mathbf{U}) = N - \mathrm{Sp}(\mathbf{D}) = N - (q - k)$.

Theorem 8. *The covariance matrix of the vector $(\tilde{\mathbf{l}}, \mathbf{z}')'$ is*

$$\Sigma_{\binom{\tilde{\mathbf{l}}}{\tilde{\mathbf{z}}}} = \begin{pmatrix} \Sigma_{\tilde{\mathbf{l}}}, & -\sigma^2\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{12} \\ -\sigma^2\mathbf{Q}_{21}\mathbf{X}\mathbf{P}^{-1}, & \Sigma_{\tilde{\mathbf{z}}} \end{pmatrix}.$$

Proof. It is sufficient to show that $M[(\tilde{\mathbf{l}} - \mathbf{y})(\tilde{\mathbf{z}} - \mathbf{z})'] = -\sigma^2\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{12}$; $\tilde{\mathbf{l}} - \mathbf{y} = (\mathbf{I} - \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X})\Delta$, $(\tilde{\mathbf{z}} - \mathbf{z})' = -\Delta'\mathbf{X}'\mathbf{Q}_{12}$; $M(\tilde{\mathbf{l}} - \mathbf{y})(\tilde{\mathbf{z}} - \mathbf{z})' =$

$= \sigma^2[-\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{12} + \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{12}]$. With regard to the relationships $\mathbf{G} = \mathbf{X}\mathbf{P}^{-1}\mathbf{X}'$ and $\mathbf{Q}_{11}\mathbf{G}\mathbf{Q}_{12} = \mathbf{O}$ from (4) the proof is obvious.

Corollary. *If* $u = f_0 + (\mathbf{f}_1' \vdots \mathbf{f}_2')\begin{pmatrix}\mathbf{y}\\ \cdots \\ \mathbf{z}\end{pmatrix}$ *and* $\tilde{u} = f_0 + \mathbf{f}_1'\tilde{\mathbf{l}} + \mathbf{f}_2'\tilde{\mathbf{z}}$, *then* $\tilde{u} \ldots N(u;$ $\sigma^2(u))$, *where* $\sigma^2(\tilde{u}) = \sigma^2(\mathbf{f}_1'\mathbf{P}^{-1}\mathbf{f}_1 - \mathbf{f}_1'\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1}\mathbf{f}_1 - 2\mathbf{f}_1'\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{12}\mathbf{f}_2 - \mathbf{f}_2'\mathbf{Q}_{22}\mathbf{f}_2)$.

Theorem 9. *The random variable* $\tilde{\mathbf{v}}'\mathbf{P}\tilde{\mathbf{v}}/\sigma^2$ *has the* χ^2*-distribution whith* $q-k$ *degrees of freedom.*

Proof. Analogously as in the proof of Theorem 2 we have: $\tilde{\mathbf{v}}'\mathbf{P}\tilde{\mathbf{v}} = (\tilde{\mathbf{v}}'\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1})\ \mathbf{P}(\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\tilde{\mathbf{v}})$. With regard to Theorems 1 and 3 $\tilde{\mathbf{v}} = -\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\Delta$, therefore $\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\tilde{\mathbf{v}} = -\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\Delta$. Since we have $\tilde{\mathbf{v}}'\mathbf{P}\tilde{\mathbf{v}} = \Delta'\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1}\mathbf{P}\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\Delta = \Delta'\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\Delta$. Let us denote $\mathbf{P}^{\frac{1}{2}}\Delta = \delta \ldots N(0; \sigma^2\mathbf{I})$ Since $\tilde{\mathbf{v}}'\mathbf{P}\tilde{\mathbf{v}} = \delta'\mathbf{U}\delta$ where $\mathbf{U}$ is a matrix from the proof of Theorem 2. If we denote $\mathbf{F}\delta = \delta \ldots N(\mathbf{O}; \sigma^2\mathbf{I})$, then $\tilde{\mathbf{v}}'\mathbf{P}\tilde{\mathbf{v}} = \gamma'\mathbf{D}\gamma$ where $h(\mathbf{D}) = \mathrm{Sp}(\mathbf{D}) = q-k$, which proves this theorem.

Theorem 10. *For a weighted sumation of a posteriori dispersions of measured values* $\sum_{i=1}^{N} p_i\sigma^2(\tilde{l}_i)$ *the following holds* $\sum_{i=1}^{N} p_i\sigma^2(\tilde{l}_i) = \sigma^2(N - q + k)$.

Proof. $\sum_{i=1}^{N} p_i\sigma^2(\tilde{l}_i) = \mathrm{Sp}(\mathbf{P}\,\Sigma_{\tilde{l}}) = \mathrm{Sp}[\mathbf{P}\sigma^2(\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1})] = \mathrm{Sp}[\sigma^2(\mathbf{I} - \mathbf{X}'\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1})] = \sigma^2[N - \mathrm{Sp}(\mathbf{X}\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11})] = \sigma^2(N - \mathrm{Sp}(\mathbf{G}\mathbf{Q}_{11})) = \sigma^2(N - q + k)$. Theorem 7, the rule of the trace of the product of matrices and results in the proof of Theorem 2 were utilized.

Theorem 11. *Let* $\mathbf{X}$ *be a matrix of the order* $q \times N$, *where* $N \geqslant q$ *(this case occurs often) and* $h(\mathbf{X}) = q$. *Then for* $\mathbf{Q}$ *from Theorem* 2

$$\begin{aligned}\mathbf{Q}_{11} &= \mathbf{G}^{-1} - \mathbf{G}^{-1}\mathbf{X}_1(\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{G}^{-1},\\ \mathbf{Q}_{12} &= \mathbf{G}^{-1}\mathbf{X}_1(\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1)^{-1},\\ \mathbf{Q}_{22} &= -(\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1)^{-1}\end{aligned}$$

Proof. With regard to Lemma 1 the matrices $\mathbf{G}$ and $\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1$ are regular. By substituting (5) into (3) and by multiplication with (2) we can confirm that the statement is true.

Corollary 1. *In this case the vector* k *has* $q - k$ *independent components.*

Proof. With regard to Lemma 1 and Theorem 3,

$$h(\mathbf{Q}_{11}\mathbf{X}) = h(\mathbf{Q}_{11}\mathbf{X}\mathbf{P}^{-1}\mathbf{X}'\mathbf{Q}_{11}) = h(\mathbf{Q}_{11}\mathbf{G}\mathbf{Q}_{11}) = h(\mathbf{Q}_{11}) = h(\mathbf{G}\mathbf{Q}_{11}) = q - k$$

(see the proof of Theorem 2).

Corollary 2. *In this case the vector* $\tilde{\mathbf{z}}$ *is regular. With regard to Theorem* 11 $\mathbf{Q}_{22}$ *is* *namely regular and therefore with regard to Theorem* 4 *we have* $h(\mathbf{Q}_{21}\mathbf{X}) =$ $= h(\mathbf{Q}_{22}) = k$.

REFERENCES

[1] Андерсон Т.: *Введение в многомерный статистический анализ*, Государственное издательство физико-математической литературы, Москва, 1963 (перевод с английского).

[2] Bjerhammar A., *Rectangular reciprocal matrices with special reference to geodetic calculation*, Bull. geodésique *20* (1951), 188—220.

[3] Линник Ю. В.: *Метод наименьших квадратов и основы математикостатистической теории обработки наблюдений*, Госиздат физико-математической литературы, Москва 1962.

[4] Stearn J. L., Richardson H., *Adjustment of conditions with parameters and error analysis*, Bull. géodésique *64* (1962), 117—135.

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