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OUTER MEASURES ON TOPOLOGICAL SPACES

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In measure theory the following theorem is known:

Let (X, ϱ) be a metric space and μ a Carathéodory outer measure. Then all open sets are μ-measurable.

There exist also various generalizations of this theorem in which the metric space is replaced by a topological space. In this paper we shall investigate the relations between topology and outer measure which make all open sets measurable.

The following condition (the disjoint closure condition) seems to be the analogue of Carathéodory's Condition:

Definition 1. *An outer measure μ defined on a topological space $(X, \mathscr{T})$ is said to have the property (α) if for any two sets $A, B \subset X$ we have $\bar{A} \cap \bar{B} = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$*

Example 1. Let $X = \{a, b\}$, $\mathscr{T} = \{\emptyset, \{a\}, \{a, b\}\}$. We define: $\mu(\emptyset) = 0$, $\mu(\{a\}) = \mu(\{b\}) = \mu(\{a, b\}) = 1$. Evidently $(X, \mathscr{T})$ is a topological space and μ an outer measure with the property (α). We see immediately that $\emptyset$ and $\{a, b\}$ are the only measurable sets so that the open set $\{a\}$ is not measurable.

From this example we deduce that the property (α) is not a sufficient condition for the measurability of all open sets in an arbitrary topological space.

Theorem 1. *Let $(X, \mathscr{T})$ be a topological space. Let μ be an outer measure with the property (α) defined on the system of all subsets of X and let:*

$$A_n \subset A_{n+1} \subset X \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

Then all open F_σ-sets are μ-measurable.

Proof. Let A be an open F_σ-set. Then $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \subset A_{n+1}$, $A_n = \bar{A}_n$. Let $C \subset A$, $D \subset A'$. Denote $C_n = C \cap A_n$. Then $C_n \subset C_{n+1}$. Now $C = \bigcup_{n=1}^{\infty} C_n$, so that $\mu(C) = \lim_{n\to\infty} \mu(C_n)$.

Evidently $\bar{C}_n \cap \bar{D} \subset \bar{A}_n \cap \bar{A}' = A_n \cap A' \subset A \cap A' = \emptyset$.

$$\mu(C_n) + \mu(D) = \mu(C_n \cup D) \leqslant \mu(C \cup D).$$

Therefore $\mu(C) + \mu(D) = \lim_{n\to\infty} \mu(C_n) + \mu(D) = \lim_{n\to\infty} \mu(C_n \cup D) \leqslant \mu(C \cup D)$.

Note. Example 1 shows that it is essential for the open set to be F_σ.

Lemma 1. *Let (X, ϱ) be a metric space and μ an outer measure defined on the system of all subsets of X. Then for every open set A there exists a non — decreasing sequence of sets $\{A_n\}_{n=1}^{\infty}$ with these properties:*

1. $\bar{A}_n \subset A$;
2. $\overline{A_{n+2} - A_{n+1}} \cap \bar{A}_n = \emptyset$;
3. $\mu(A - \bigcup_{n=1}^{\infty} A_n) = 0$.

Proof. If $A = X$ then put $A_n = X$. Let $A \neq X$. Then $B = X - A \neq \emptyset$. Define $A_n = \left\{x : x \in X;\ \varrho(B, x) < \frac{1}{n}\right\}$. Evidently $\{A_n\}_{n=1}^{\infty}$ is the required sequence of sets.

Lemma 2. *Let $(X, \mathscr{T})$ be a topological space. Let μ be an outer measure with the property (α) defined on the system of all subsets of X. Let A be an open set and $C \subset A$. Suppose that there exists a nondecreasing sequence of sets $\{A_n\}_{n=1}^{\infty}$ with the following properties:*

1. $\bar{A}_n \subset A$;
2. $\overline{A_{n+2} - A_{n+1}} \cap \bar{A}_n = \emptyset$;
3. $\mu(A - \bigcup_{n=1}^{\infty} A_n) = 0$.

Then there exists a nondecreasing sequence of sets $\{M_n\}_{n=1}^{\infty}$ such that

1. $\bar{M}_n \subset A$;
2. $\lim_{n\to\infty} \mu(M_n) = \mu(C)$.

Proof. Let $A \subset X$ be an open set. Let $C \subset A$. Let $\{A_n\}_{n=1}^{\infty}$ be the sequence of sets mentioned in the hypothesis and $M = C \cap \bigcup_{n=1}^{\infty} A_n$. Evidently $M \subset C$, so that $\mu(M) \leqslant \mu(C)$. Now $C - M \subset A - \bigcup_{n=1}^{\infty} A_n$, so that $\mu(M) = \mu(C)$.

Let $M_n = C \cap A_n$. Clearly $M_n \subset M_{n+1}$ and $\bar{M}_n \subset \bar{A}_n \subset A$.

Moreover, $\overline{M_{n+2} - M_{n+1}} \cap \bar{M}_n = \overline{(M \cap A_{n+2}) - (M \cap A_{n+1})} \cap \overline{(M \cap A_n)} = = \overline{M \cap (A_{n+2} - A_{n+1})} \cap \overline{(M \cap A_n)} \subset \overline{A_{n+2} - A_{n+1}} \cap \bar{A}_n = \emptyset$. Therefore $\overline{M_{n+2} - M_{n+1}} \cap \cap \bar{M}_n = \emptyset$.

Furthermore, $M \subset \bigcup_{n=1}^{\infty} A_n \Rightarrow M = M \cap \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (M \cap A_n) = \bigcup_{n=1}^{\infty} M_n$, so that $\mu(M - \bigcup_{n=1}^{\infty} M_n) = \mu(\emptyset) = 0$. As $M \supset M_n$, we have $\mu(M) \geqslant \mu(M_n)$ and therefore

$$(1) \qquad \mu(M) \geqslant \lim_{n\to\infty} \mu(M_n) .$$

Let $P_n = M_{n+1} - M_n$. Then $M = M_{2n} \cup \bigcup_{i=n}^{\infty} P_{2i} \cup \bigcup_{i=n}^{\infty} P_{2i+1}$, so that

$$(2) \qquad \mu(M) \leqslant \mu(M_{2n}) + \sum_{i=n}^{\infty} \mu(P_{2i}) + \sum_{i=n}^{\infty} \mu(P_{2i+1}) .$$

This gives us $\mu(M) \leqslant \lim_{n\to\infty} \mu(M_{2n}) + \lim_{n\to\infty} \sum_{i=n}^{\infty} \mu(P_{2i}) + \lim_{n\to\infty} \sum_{i=n}^{\infty} \mu(P_{2i+1})$.

If both infinite series in (2) are convergent, their limits are zero, so that $\mu(M) \leqslant \lim_{n\to\infty} \mu(M_{2n}) = \lim_{n\to\infty} \mu(M_n)$.
Owing to (1), we have $\mu(M) = \lim_{n\to\infty} \mu(M_n)$.
Now let one of the series in (2) be divergent, for instance the first one We have

$$\overline{P_{2i+2}} \cap \bar{P}_{2i} \subset \overline{M_{2i+3} - M_{2i+2}} \cap \overline{M_{2i+1}} = \emptyset .$$

Using the property (α), we have

$$\sum_{i=1}^{n-1} \mu(P_{2i}) = \mu(\bigcup_{i=1}^{n-1} P_{2i}) \leqslant \mu(\bigcup_{i=1}^{2n-1} P_i) \leqslant \mu(M_{2n}) .$$

This yields $\infty = \lim_{n\to\infty} \sum_{i=1}^{n-1} \mu(P_{2i}) \leqslant \lim_{n\to\infty} \mu(M_{2n}) \leqslant \mu(M)$, so that

$$\lim_{n\to\infty} \mu(M_n) = \mu(M).$$

In case that the second infinite series is divergent, the method of the proof is analogous.

This proves that the sequence $\{M_n\}_{n=1}^{\infty}$ is the sequence of sets postulated by our lemma.

Theorem 2. *Let $(X, \mathscr{T})$ be a topological space, μ an outer measure defined on the system of all subsets of X and having the property (α), A an open set. Suppose that there exists a nondecreasing sequence of sets $\{A_n\}_{n=1}^{\infty}$ with the properties:*

1. $\bar{A}_n \subset A$;

2. $\overline{A_{n+2} - A_{n+1}} \cap \bar{A}_n = \emptyset$;

3. $\mu(A - \bigcup_{n=1}^{\infty} A_n) = 0$.

Then the set A is μ-measurable.

Proof. Let $C \subset A$, $D \subset A'$. According to Lemma 2 we can form the sequence $\{M_n\}_{n=1}^{\infty}$ corresponding to C. Using the property (α) we have

$$\mu(D) + \mu(M_n) = \mu(M_n \cup D) \leqslant \mu(C \cup D)$$

Then

$$\mu(C) + \mu(D) = \lim_{n\to\infty} \mu(M_n) + \mu(D) \leqslant \mu(C \cup D),$$

proving the μ-measurability of A.

A corollary of Theorem 2 is

Theorem 3. *Let $(X, \mathcal{T})$ be a topological space, μ an outer measure defined on the system of all subsets of X and having the property (α), f a continuous function on X.* Then the set $\{x : x \in X; f(x) = 0\}$ *is μ-measurable.*

Proof. Let $B = \{x : x \in X; f(x) = 0\}$. Evidently B is μ-measurable if and only if the set $A = B'$ is. Clearly A is an open set. Let

$$A_n = X - \left\{x : x \in X;\, |f(x)| < \frac{1}{n}\right\}$$

The sets A, A_n satisfy the conditions of Theorem 2, therefore A is μ-measurable.

Corollary. *In a topological space the property (α) ensures the μ-measurability of all continuous functions.*

Proof. Let $f(x)$ be a continuous function. Let c be a real number. According to theorem 3 it is enough to find a continuous function $g(x)$, so that

$$\{x : x \in X;\ f(x) \leqslant c\} = \{x : x \in X;\ g(x) = 0\}.$$

Such a function is for example

$$g(x) = (f(x) - c)^+.$$

A question arises whether the property (α) is not a necessary condition as well. The following example answers this question in the negative.

Example. Let $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$. Then $(X, \mathcal{T})$ is a topological space on which every continuous function is constant. We choose the following function μ:

$$\mu(\emptyset) = 0;\ M \subset X,\ M \neq \emptyset \Rightarrow \mu(M) = 1$$

Now μ is an outer measure and all continuous functions are μ-measurable. But

$$\overline{\{a\}} \cap \overline{\{b\}} = \{a\} \cap \{b\} = \emptyset;\ \mu(\{a\}) + \mu(\{b\}) = 1 + 1 \neq 1 = \mu(\{a, b\}),$$

so that μ does not have the property (α).

It is easy to see that changing the formulation of Theorem 2 would give us a sufficient condition for the μ-measurability of all open sets of a topological space $(X, \mathscr{T})$. In the following we shall give a necessary and sufficient condition for this fact.

Theorem 4. *Let $(X, \mathscr{T})$ be a topological space, μ an outer measure defined on the system of all subsets of X. All open sets are μ-measurable if and only if μ satisfies the condition:*

$$(\bar{A} \cap B = \emptyset \text{ or } A \cap \bar{B} = \emptyset) \Rightarrow \mu(A \cup B) \geqslant \mu(A) + \mu(B).$$

Proof. 1) Let $A \subset X$ be an open set, $C \subset A$, $D \subset A'$. Then $C \cap \bar{D} \subset A \cap \bar{A}' = \emptyset$, so that

$$\mu(C \cup D) \geqslant \mu(C) + \mu(D)$$

by hypothesis.

2) Let all open sets be μ-measurable. Let there exist two sets A, B such that $\bar{A} \cap B = \emptyset$ and

$$\mu(A \cup B) < \mu(A) + \mu(B).$$

Let $C = \bar{A}$, $D = C'$. Then the open set D is not μ-measurable, which is a contradiction.

In case that $A \cap \bar{B} = \emptyset$, we put $C = \bar{B}$ and continue as before.

Corollary. *In metric spaces the following conditions are equivalent:*

(1) μ is a Carathéodory outer measure;

(2) μ is an outer measure having the property (α);

(3) μ is an outer measure satisfying the condition

$$(\bar{A} \cap B = \emptyset \text{ or } A \cap \bar{B} = \emptyset) \Rightarrow \mu(A \cup B) \geqslant \mu(A) + \mu(B).$$

Proof. Evidently (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Let us prove that (3) $\Rightarrow$ (1). If (3) holds, then according to Theorem 4 all open sets are measurable. A method of proof analogical to the second part of the proof of Theorem 4 shows that in that case μ is also a Carathéodory outer measure.

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