Matematický časopis

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Matematický časopis, Vol. 19 (1969), No. 4, 249--251

Persistent URL: `http://dml.cz/dmlcz/126661`

MATEMATICKÝ ČASOPIS
ROČNÍK 9 1969 ČÍSLO 4

LUSIN'S THEOREM IN AN ABSTRACT SPACE

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In [3] W. J. Trjitzinsky obtained a generalization of Lusin's theorem (hereafter called the L—T—Th.) for necessary and sufficient conditions for measurability of a finite real valued function on a set with finite measure. Also in [3] (17.11) Trjitzinsky proved a Lusin type theorem for sets with infinite measures. In this paper another Lusin type theorem is obtained for sets with infinite measures. It is easy to give an example that all assumptions of this theorem but not all assumptions of cited Trjitzinsky's theorem are satisfied.

Definition 1. *Let F be any family of sets. A point x is indefinitely covered in the sense of measure Φ by the family F, if F contains a sequence $\{E_n\}$ of sets such that $x \in E_n$ for all n and the exterior measures of the E_n approach zero, i. e. $\Phi_e(E_n) \to 0$.* [1] *A set all of whose points satisfy the above conditions is said to be a set indefinitely covered by F and is denoted by $I(F)$.*

Definition 2. *Let X be a subset of $I(F)$; then $F(X)$ is the family of all E of F for which $X \cap E \neq \emptyset$.*

Definition 3. ([3], Def. 11.5) *A family F of sets is a simply regular family if the following are satisfied:*

i) *E is Φ-measurable and $0 < \Phi(E) < \infty$ for all $E \in F$,*

ii) *$F = \bigcup_{n=1}^{\infty} F_n$ where $F_n \subset F_{n+1}$ $(n = 1, 2, 3, \ldots)$,*

iii) *$\Phi(\bigcup_{E \in F_n} E) < \infty$, $\cup\{E : E \in G\}$ is Φ-measurable and $\Phi(\cup\{E : E \in G\}) < \infty$ for every $G \subset F_n$.*

Definition 4. ([3], Def. 12.5) *Let F be a simply regular family of sets. If X is a subset of $I(F_n)$ we say that X is a noyau with respect to F_n if $\Phi\,(C_n(X)) = 0$ where $C_n(X) = I(F_n(X)) - X$.*

If X is a subset of $I(F)$, X is a noyau with respect to F if $\Phi(C_n(X \cap I(F_n))) = 0$ i. e. if $X \cap I(F_n)$ is a noyau with respect to F_n.

Note. Every noyau X is measurable and the union of a finite number of noyaux is a noyau.

Definition 5. ([3], Def. 12.8). *A family F of sets is a completely regular family (C. R.) if F is simply regular and if to every set X contained in $I(F_n)$ and measurable and to every $\varepsilon > 0$ there corresponds a noyau Y with respect to F_n, such that $Y \subset X$ and $\Phi(X - Y) < \varepsilon$.*

Definition 6. ([3], p. 86). *Let f be a real valued function defined on H such that for every real number c the sets*

$H_c^+ = \{x \in H : f(x) \geqslant c\}$,
$H_c^- = \{x \in H : f(x) \leqslant c\}$ *are noyaux with respect to F. We say that f is a pseudocontinuous function (f is P—C—F).*

Definition 7. *Let F be a simply regular family of sets. We say F fulfills the condition (C), if and only if every noyau with respect to F_n is a noyau with respect to F_{n+1} ($n = 1, 2, \ldots$).*

L-T-Theorem. ([3], Theorem 17.3). *Let F be C.R. with $\Phi(\bigcup_{E \in F} E) < \infty$. Let H be a measurable subset of $I(F)$ and assume that f is a finite real valued function defined on H. In order that f be measurable on H it is necessary and sufficient that to every real number $\varepsilon > 0$ there correspond a noyau N contained in H, on which f is P—C—F and $\Phi(H—N) < \varepsilon$.*

The main result of this paper is obtained in the following theorem.

Theorem. *Let F be C.R. and fulfil the condition (C). Let H be a measurable subset of $\bigcup_{n=1}^{\infty} I(F_n)$ and f be a finite real valued function on H. Then f is measurable if and only if to every $\varepsilon > 0$ there corresponds a noyau contained in H, on which f is P—C—F and $\Phi(H—N) < \varepsilon$.*

Proof. Assume that f is measurable. Put $H_n = H \cap I(F_n) - H \cap I(F_{n-1})$, $n = 2, 3, \ldots, H_1 = I(F_1)$, By L—T-Th. to any $\varepsilon > 0$ and any n there is a noyau N_n with respect to F_n such that $\Phi(H_n - N_n) < \dfrac{\varepsilon}{2^{n+1}}$ and f is P—C—F_n on N_n.

Put $N = \bigcup_{n=1}^{\infty} N_n$. We prove that N is a noyau. Since F fulfils (C), N_i is noyau with respect to F_n for $i = 1, 2, \ldots, n$. Hence $\Phi(I(F_n(N \cap I(F_n))) - N \cap I(F_n)) = \Phi(I(F_n(\bigcup_{i=1}^{n} N_i)) - \bigcup_{i=1}^{n} N_i) \leqq \sum_{i=1}^{n} \Phi(I(F_n(N_i)) - N_i) = 0.$

We get $N \cap I(F_n)$ is a noyau with respect to F_n, therefore N is a noyau with respect to F. Clearly $\Phi(H - N) \leqq \sum_{n=1}^{\infty} \Phi(H_n - N_n) < \varepsilon$ and f is P—C—F on N.

If f fulfils the above condition, then to any n there is a noyau N_n such that f is P—C—F on N_n and $\Phi(H - N_n) < \frac{1}{n}$. Since every noyau is measurable, f is measurable on N_n, hence on $\bigcup_{n=1}^{\infty} N_n$ too. But $\Phi(H - \bigcup_{n=1}^{\infty} N_n) = 0$, therefore f is measurable on H.

This paper is based on a doctoral dissertation written at the University of Illinois under the supervision of Professor W. J. Trjitzinsky, to whom the author is greatly indebted. He is also grateful for the helpful suggestions of the reviewer, in particular for the elegant formulation of Definition 7.

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Received October 4, 1967.

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