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INVOLUTED RESTRICTIVE BISEMIGROUPS OF BINARY RELATIONS

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Let ϱ and σ be binary relations between the elements of two sets A and B. Denote by $\mathrm{pr}_1\varrho$ the first projection of ϱ and by $\mathrm{pr}_2\sigma$ the second projection of σ. Define $\varrho \rhd \sigma = (\mathrm{pr}_1\varrho \times B) \cap \sigma$, $\varrho \lhd \sigma = \varrho \cap (A \times \mathrm{pr}_2\sigma)$. Then $\varrho \rhd \sigma$ and $\varrho \lhd \sigma$ are binary relations between the elements of the same sets A and B. The operations $\rhd$ and $\lhd$ are called *the restrictive multiplication of the first kind* (or *the left restrictive multiplication*) and *the restrictive multiplication of the second kind* (or *the right restrictive multiplication*). A *left restrictive semigroup of binary relations is* any pair $(\Phi, \rhd)$ where Φ is a non empty set of binary relations between elements of some two sets, closed under $\rhd$. *Right restrictive semigroups of binary relations* are defined analogously. An algebra $(\Phi, \rhd, \lhd)$ where $(\Phi, \rhd)$ is the left and $(\Phi, \lhd)$ the right restrictive semigroup of binary relations is called a *restrictive bisemigroup of binary relations.* Restrictive products of both kinds of two many-to-one (or one-to-one) binary relations are many-to-one (or one-to-one) binary relations, hence one may speak of restrictive semigroups and bisemigroups of (partial) mappings and one-to-one (partial) mappings.

An algebra isomorphic to a (left, right) restrictive semigroup of binary relations is called a *(left, right) restrictive semigroup.* An algebra isomorphic to a restrictive bisemigroup of binary relations is called a *restrictive bisemigroup.*

These definitions are due to V. V. Wagner [1]. Restrictive bisemigroups were introduced by the author [2], but the idea goes back to V. V. Wagner [3].

V. V. Wagner [1] proved that left restrictive semigroups are characterized by the following three identities:

$$(1) \qquad (x \rhd y) \rhd z = x \rhd (y \rhd z) \ \textit{(associativity)}$$

$$(2) \qquad x \rhd x = x \ \textit{(idempotency)}$$

$$(3) \qquad x \rhd y \rhd z = y \rhd x \rhd z \ \textit{(left pseudocommutativity)}$$

Operations satisfying (3) are also called *right normal.*

V. V. Wagner [1] also proved that every left restrictive semigroup is isomorphic to a left restrictive semigroup of partial mappings. The author [2] proved that every left restrictive semigroup is isomorphic to a left restrictive semigroup of one-to-one partial mappings, and is isomorphic to a left restrictive semigroup of partial mappings of a set *onto* another set [4].

Dually, right restrictive semigroups are characterized by the identities:

(4) $$(x \lhd y) \lhd z = x \lhd (y \lhd z)\,,$$

(5) $$x \lhd x = x\,,$$

(6) $$x \lhd y \lhd z = x \lhd z \lhd y\,.$$

The author [2] proved that restrictive bisemigroups are characterized by the identities (1)—(6) and the identity:

(7) $(x \rhd y) \lhd z = x \rhd (y \lhd z)$ (*associativity of the pair* $(\rhd, \lhd)$)

In what follows the product $(x \rhd y) \lhd z$ will be denoted by $x \rhd y \lhd z$.

In view of possible applications the most interesting class of restrictive bisemigroups is the class of restrictive bisemigroups of one-to-one partial mappings. If V is a differentiable manifold and K is its coordinate atlas (i. e., the set of all admissible coordinate systems) then K is closed under $\rhd$ and $\lhd$ and $(K, \rhd, \lhd)$ is a restrictive bisemigroup. The present author has proved (unpublished) that two differentiable manifolds with isomorphic restrictive bisemigroups of coordinate systems are isomorphic (i. e., diffeomorphic). Hence, a rather complex structure of a differentiable manifold may be completely defined by two rather trivial idempotent associative algebraic operations.

The author [4] proved that a restrictive bisemigroup is isomorphic to a restrictive bisemigroup of partial mappings if and only if it satisfies the identity:

(8) $$(x \lhd y) \rhd x = x \lhd y$$

and is isomorphic to a restrictive bisemigroup of one-to-one partial mappings if and only if it satisfies the identities (8) and

(9) $$x \lhd (y \rhd x) = y \rhd x\,.$$

Every restrictive bisemigroup $(\Phi, \rhd, \lhd)$ of binary relations is ordered by the inclusion relation $\subset$. An ordered restrictive bisemigroup $(A, \rhd, \lhd, \leqslant)$ is isomorphic to a restrictive bisemigroup of binary relations ordered by the inclusion relation if and only if the order $\leqslant$ satisfies the following conditions:

(10) $$v \leqslant w,\ x \leqslant y \ \rightarrow \ v \rhd x \leqslant w \rhd y\,,$$

(*stability of* $\leqslant$)

$$v \leqslant w,\ x \leqslant y \ \rightarrow\ v \lhd x \leqslant w \lhd y\,, \tag{11}$$

$$x \rhd y \leqslant y\,, \tag{12}$$

$$x \lhd y \leqslant x\,, \tag{13}$$

$$x \leqslant y \ \rightarrow\ y \rhd x \lhd y = x\,. \tag{14}$$

This was proved in [4].

The present note is devoted to restrictive bisemigroups of binary relations between the elements of two equal sets (the so-called *homogeneous* binary relations). If $\sigma \subset A \times A$ is a binary relation between the elements of a set A then ϱ^{-1} denotes *the converse* of ϱ: $(a_1, a_2) \in \varrho^{-1} \leftrightarrow (a_2, a_1) \in \varrho$.

An *involuted set of binary relations* is any pair $(\Phi, {}^{-1})$ where Φ is a non-empty set of homogeneous binary relations closed under the operation ${}^{-1}$ of conversion. An *involuted bisemigroup of binary relations* is an algebra $(\Phi, \rhd, \lhd, {}^{-1})$ where $(\Phi, \rhd, \lhd)$ is a restrictive bisemigroup of binary relations and $(\Phi, {}^{-1})$ is an involuted set of binary relations. We consider also *ordered involuted restrictive bisemigroups of binary relations*, i. e., algebraic systems $(\Phi, \rhd, \lhd, {}^{-1}, \subset)$ where $(\Phi, \rhd, \lhd, {}^{-1})$ is an involuted restrictive bisemigroup of binary relations and $(\Phi, \subset)$ is a set of binary relations ordered by the inclusion relation.

Let x and y denote binary relations, x^{-1} and y^{-1} their converses and $\leqslant$ denote the inclusion relation. It is easy to verify the following properties:

$$x \leqslant y \ \rightarrow\ x^{-1} \leqslant y^{-1}, \ (\textit{stability of } \leqslant \textit{ under conversion}) \tag{15}$$

$$(x^{-1})^{-1} = x\,,\ (\textit{involutivity}) \tag{16}$$

$$(x \rhd y)^{-1} = y^{-1} \lhd x^{-1}. \tag{17}$$

The identities (16) and (17) imply

$$x \rhd y = (y^{-1} \lhd x^{-1})^{-1}, \tag{18}$$

$$x \lhd y = (y^{-1} \rhd x^{-1})^{-1} \tag{19}$$

i. e., each one of the restrictive operations is a derived operation from the other one and from conversion ${}^{-1}$. Identities (18) and (2), (5) imply both (16) and (17). If (16) and (17) are satisfied, then (1) is equivalent to (4), (2) to (5), (3) to (6), (8) to (9). If (15)—(17) are satisfied, then (10) is equivalent to (11), (12) to (13). If (1), (2), (7), (15)—(17) are satisfied, then (14) is equivalent to each of the formulas:

$$x \leqslant y \ \rightarrow\ y \rhd x = x\,, \tag{20}$$

$$x \leqslant y \ \rightarrow\ x \lhd y = x\,. \tag{21}$$

Let (1), (2) and (7) lbe satisfied. If (14) is true and $x \leqslant y$ then $y \rhd x = y \rhd (y \rhd x \lhd y) = (y \rhd y) \rhd x \lhd y = y \rhd x \lhd y = x$. If (16) and (17) are satisfied and (20) is true then $x \leqslant y$ imply $y \rhd x = x$ or, equivalently, providing (15), $x \leqslant y$ imply $x^{-1} \leqslant y^{-1}$, which imply $y^{-1} \rhd x^{-1} = x^{-1}$. Hence $x \lhd y = (y^{-1} \rhd x^{-1})^{-1} = (x^{-1})^{-1} = x$. Let (15)—(17), (21) be satisfied and $x \leqslant y$. Then $x \lhd y = x$ and $x^{-1} \leqslant y^{-1}$, hence $y \rhd x = (x^{-1} \lhd y^{-1})^{-1} = (x^{-1})^{-1} = x$ and $y \rhd x \lhd y = y \rhd (x \lhd y) = y \rhd x = x$.

Let $(A, \rhd, \lhd)$ be a restrictive bisemigroup. Define a binary relation α between the elements of A by the formula:

$$(a_1, a_2) \in \alpha \quad \leftrightarrow \quad a_1 \rhd a_2 \lhd a_1 = a_1 \,. \tag{22}$$

It is known [2, 4] that α is an order relation. It is called *the canonical order relation* of the restrictive bisemigroup. This order relation satisfies the conditions (10)—(14). If $(A, \rhd, \lhd)$ satisfies (8) or (9), then α is the only order relation satisfying (10)—(14).

Our main result is the following

Theorem of representation. *Every algebra* $(A, \rhd, \lhd, {}^{-1})$ *satisfying the identities* (1)—(3), (7), (16), (17) *is isomorphic to an involuted restrictive bisemigroup of binary relations. If this algebra satisfies also* (8) *then it is isomorphic to an involuted restrictive bisemigroup of one-to-one partial transformations.*

Every algebraic system $(A, \rhd, \lhd, {}^{-1}, \leqslant)$, *where* $\leqslant$ *is an order relation, is isomorphic to an ordered involuted restrictive bisemigroup of binary relations, if this system satisfies* (1)—(3), (7), (10), (12), (15)—(17), (20).

Proof. Let $(A, \rhd, \lhd, {}^{-1}, \leqslant)$ satisfy (1)—(3), (7), (10), (12), (15)—(17), (20). Then it satisfies (1)—(7), (10)—(21). Let $F(A)$ be a free group over the alphabet A such that for every $a \in A$ a^{-1} is the group inverse of a. Let R be any faithful simply transitive representation of $F(A)$ by permutations of some set B (say, let R be the representation of $F(A)$ by translations).

Clearly, $(A, \rhd, \lhd)$ is a restrictive bisemigroup. Let ε_1 and ε_2 be its canonical equivalence relations [2, 4], i. e., $a_1 \equiv a_2(\varepsilon_1) \leftrightarrow a_1 \rhd a_2 = a_2$ and $a_2 \rhd a_1 = a_1$; $a_1 \equiv a_2(\varepsilon_2) \leftrightarrow a_1 \lhd a_2 = a_1$ and $a_2 \lhd a_1 = a_2$. The identities (16) and (17) imply that $a_1 \equiv a_2(\varepsilon_1) \leftrightarrow a_1^{-1} \equiv a_2^{-1}\,(\varepsilon_2)$. In what follows we write ε instead of ε_1. $\varepsilon\langle a\rangle$ denotes the ε-class containing a.

Let $\Theta_{\varepsilon(a)}$ be a one-to-one mapping of B onto some set $\mathfrak{b}_{\varepsilon\langle a\rangle}$ and sets $\mathfrak{b}_{\varepsilon\langle a_1\rangle}$ and $\mathfrak{b}_{\varepsilon\langle a_2\rangle}$ be disjoint if $\varepsilon\langle a_1\rangle \neq \varepsilon\langle a_2\rangle$. Denote $C = \bigcup_{a\in A}\mathfrak{b}_{\varepsilon\langle a\rangle}$. Define $\varphi_a = \Theta_{\varepsilon\langle a^{-1}\rangle} \circ R(a) \circ \Theta^{-1}_{\varepsilon\langle a\rangle}$, $P(a) = \bigcup_{b\leq a} \varphi_b$.

Then P is an isomorphism of $(A, \rhd, \lhd, {}^{-1}, \leqslant)$ onto some ordered involutive restrictive bisemigroup of binary relations between the elements of C, i. e., the following formulas hold:

$$(23) \qquad P(a_1) \rhd P(a_2) = P(a_1 \rhd a_2)\,,$$

$$(24) \qquad P(a_1) \lhd P(a_2) = P(a_1 \lhd a_2)\,,$$

$$(25) \qquad \frac{-1}{P(\mathrm{a})} = P(a^{-1})\,,$$

$$(26) \qquad P(a_1) \subset P(a_2) \quad \leftrightarrow \quad a_1 \leqslant a_2\,.$$

If (26) holds and $P(a_1) = P(a_2)$, then $P(a_1) \subset P(a_2)$, $P(a_2) \subset P(a_1)$. Hence $a_1 \leqslant a_2$ and $a_2 \leqslant a_1$, i. e. $a_1 = a_2$ and P is one-to-one.

For the proof of (23) we need the formula:

$$(27) \qquad b \leqslant a_1 \rhd a_2 \quad \leftrightarrow \quad (\bigvee a)\,[b \equiv a(\varepsilon) \wedge a \leqslant a_1 \wedge b \leqslant a_2]\,.$$

Here $\bigvee$ is the existence quantifier and $\wedge$ is the conjunction.

Let $b \leqslant a_1 \rhd a_2$. Denote $a = b \rhd a_1$. By (12), $a \leqslant a_1$ and $b \leqslant a_1 \rhd a_2 \leqslant a_2$. By (1) and (2), $b \rhd a = b \rhd b \rhd a_1 = b \rhd a_1 = a$. By (1)—(3) and (20), $a \rhd b = b \rhd a_1 \rhd b = a_1 \rhd b \rhd b = a_1 \rhd b = a_1 \rhd ((a_1 \rhd a_2) \rhd b)$ $= (a_1 \rhd a_1) \rhd a_2 \rhd b = (a_1 \rhd a_2) \rhd b = b$, hence, $b \equiv a(\varepsilon)$.

Let now $(\bigvee a)[b \equiv a(\varepsilon) \wedge a \leqslant a_1 \wedge b \leqslant a_2]$. By (10), $b = a \rhd b \leqslant a_1 \rhd b \leqslant a_1 \rhd a_2$, i. e., $b \leqslant a_1 \rhd a_2$. We have proved (27).

$P(a_1) \rhd P(a_2) = (\bigcup_{a \leq a_1} \varphi_a) \rhd (\bigcup_{b \leq a_2} \varphi_b) = \bigcup_{a \leq a_1,\, b \leq a_2} \varphi_a \rhd \varphi_b =$ $= \bigcup_{a \equiv b(\varepsilon),\, a \leq a_1,\, b \leq a_2} \varphi_a \rhd \varphi_b = \bigcup_{b \leq a_1 \rhd a_2} \varphi_b = P(a_1 \rhd a_2)$. In the proof we used the following property: $\varphi_a \rhd \varphi_b = \emptyset$ if $\varepsilon\langle a\rangle \neq \varepsilon\langle b\rangle$, and $\varphi_a \rhd \varphi_b = \varphi_b$ otherwise.

The proof of (24) is quite analogous. It may also be obtained from (23) and (25). Using (19) as an axiom and as a property of binary relations we obtain $P(a_1) \rhd P(a_2) = \dfrac{-1}{\dfrac{-1}{P(a_2)} \rhd \dfrac{-1}{P(a_1)}} = \dfrac{-1}{P(a_2^{-1}) \rhd P(a_1^{-1})} = \dfrac{-1}{P(a_2^{-1} \rhd a_1^{-1})} =$

$$= P((a_2^{-1} \rhd a_1^{-1})^{-1}) = P(a_1 \lhd a_2)\,.$$

$\varphi_a^{-1} = \Theta_{\varepsilon\langle a\rangle} \circ \dfrac{-1}{R(a)} \circ \Theta^{-1}_{\varepsilon\langle a^{-1}\rangle} = \Theta_{\varepsilon\langle a\rangle} \circ R(a^{-1}) \circ \Theta^{-1}_{\varepsilon\langle a^{-1}\rangle} = \varphi_{a^{-1}}$. Hence, $\dfrac{-1}{P(a)} =$

$$= \frac{-1}{\bigcup_{b \leq a} \varphi_b} = \bigcup_{b \leq a} \varphi_b^{-1} = \bigcup_{b^{-1} \leq a^{-1}} \varphi_{b^{-1}} = P(a^{-1}).$$

Let $P(a_1) \subset P(a_2)$. Then $\varphi_{a_1} \subset P(a_1) \subset P(a_2)$, therefore $\varphi_{a_1} \cap \varphi_b \neq \emptyset$ for some $b \leqslant a_2$. It means that $R(a_1) \cap R(b) \neq \emptyset$.(1) By simple transitivity of R,

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(1) The proof of this fact may be found in [4], p. 118.

$R(a_1) = R(b)$ and $a_1 = b$. Hence $a_1 \leqslant a_2$. If $a_1 \leqslant a_2$ then evidently $P(a_1) \subset \subset P(a_2)$.

If $(A, \triangleright, \triangleleft, ^{-1})$ is an algebra satisfying (1)—(3) (7), (16), (17), then $(A, \triangleright, \triangleleft, ^{-1}, \alpha)$ satisfies (1)—(3), (7), (10), (12), (15)—(17), (20) and is isomorphic to an ordered involuted restrictive bisemigroup of binary relations. If (8) is also satisfied, then the isomorphism P constructed as above is an isomorphism of $(A, \triangleright, \triangleleft, ^{-1}, \alpha)$ onto a bisemigroup of one-to-one partial transformations. If (8) is satisfied, then (9) is also satisfied. Let $(c, c_1) \in P(a)$ and $(c, c_2) \in P(a)$. It means that $(c, c_1) \in \varphi_{b_1}$ and $(c, c_2) \in \varphi_{b_2}$ for some $b_i \leqslant a$ $(i = 1, 2)$. We see that $\mathrm{pr}_1\, \varphi_{b_1} \cap \mathrm{pr}_1\, \varphi_{b_2} \neq \emptyset$. It is possible only if $\varepsilon\langle b_1\rangle = \varepsilon\langle b_2\rangle$. Therefore, $b_i \triangleright b_j = b_j$ for $i, j = 1, 2$. The order $\leqslant$ coincides with α, hence $b_i \triangleright a \triangleleft b_i = = b_i$. Using (8) we have: $b_1 = b_1 \triangleright (a \triangleleft b_1) = b_1 \triangleright ((a \triangleleft b_1) \triangleright a) = (b_1 \triangleright a \triangleleft b_1) \triangleright a = b_1 \triangleright a = (b_2 \triangleright b_1) \triangleright a = b_1 \triangleright (b_2 \triangleright a) = b_1 \triangleright b_2 = b_2$. We used the equality $b_2 \triangleright a = b_2$ which may be proved exactly as the equality $b_1 \triangleright a = b_1$. Hence, $\varphi_{b_1} = \varphi_{b_2}$. But φ_{b_1} is one-to-one, hence $c_1 = c_2$. In the same way we prove that $(c_1, c) \in P(a)$ and $(c_2, c) \in P(a)$ imply $c_1 = c_2$. It means that $P(a)$ is one-to-one for all $a \in A$. Therefore, if $(A, \triangleright, \triangleleft, ^{-1})$ satisfies (1)—(3), (7), (16), (17) (and (8)) then this algebra is isomorphic to an involuted restrictive bisemigroup of (one-to-one) binary relations, which completes the proof of our theorem.

As an important example of an involuted restrictive bisemigroup satisfying (8) we give the following one: let G be any inverse semigroup (= generalized group, or pseudogroup, or groupoid, or paragroup) g^{-1} be the inverse for $g \in G$. Define $g_1 \triangleright g_2 = g_1 g_1^{-1} g_2$ and $g_1 \triangleleft g_2 = g_1 g_2^{-1} g_2$. Then $(G, \triangleright, \triangleleft, ^{-1})$ is an involuted restrictive bisemigroup satisfying (8).

Here and henceforward an algebra satisfying (1)—(3), (7), (16), (17) is called an *involuted restrictive bisemigroup*, and an algebraic system satisfying (1)—(3), (7), (10), (12), (15)—(17) is called an *ordered involutive restrictive bisemigroup*.

We say that a restrictive bisemigroup $(A, \triangleright, \triangleleft)$ *admits involution* if $(A, \triangleright, \triangleleft, ^{-1})$ is an involuted restrictive bisemigroup for some unary operation $^{-1}$. Clearly, $(A, \triangleright, \triangleleft)$ admits involution if and only if there exists an involutive antiisomorphism of the left restrictive semigroup $(A, \triangleright)$ onto the right restrictive semigroup $(A, \triangleleft)$ (a transformation φ is called *involutive* if $\varphi \circ \varphi$ is an identical transformation).

An algebra $(A, \triangleright, ^{-1})$ is called an *involuted left restrictive semigroup* if it is isomorphic to an algebra $(\Phi, \triangleright, ^{-1})$ where $(\Phi, \triangleright)$ is a left restrictive semigroup of binary relations and $(\Phi, ^{-1})$ is an involuted set of binary relations. Our Theorem of representation yields the following

Corollary. *An algebra* $(A, \rhd, ^{-1})$ *is an involuted left restrictive semigroup if and only if it satisfies* (1)—(3), (16) *and the following identity:*

$$(x \rhd (y \rhd z)^{-1})^{-1} = y \rhd (x \rhd z^{-1})^{-1}. \tag{28}$$

Proof. Clearly $(A, \rhd, ^{-1})$ is an involuted left restrictive semigroup if and only if there exists a binary operation $\lhd$ on A such that $(A, \rhd, \lhd, ^{-1})$ is an involuted restrictive bisemigroup, i. e., it satisfies (1)—(3), (7), (16) and (17). From (17) we deduce that $x \lhd y = (y^{-1} \rhd x^{-1})^{-1}$. The identity (7) may be written in the form $(y \rhd z) \lhd x^{-1} = y \rhd (z \lhd x^{-1})$. This form is equivalent to the identity (28).

The identity (9) is equivalent to

$$(x \rhd y)^{-1} \rhd y^{-1} = (x \rhd y)^{-1}. \tag{29}$$

This identity characterizes involuted left restrictive semigroups isomorphic to involuted left restrictive semigroups of one-to-one partial transformations.

In a quite analogous way one can define and characterize involuted right restrictive semigroups.

We will consider now restrictive bisemigroups and semigroups of certain special binary relations.

Let ϱ be a binary relation between the elements of a set A. If $B \subset A$, then Δ_B denotes the identical binary relation over B, i. e., $(a_1, a_2) \in \Delta_B \leftrightarrow a_1 = a_2 \in B$. Let $\mathrm{pr}\, \varrho = \mathrm{pr}_1\, \varrho \cup \mathrm{pr}_2\, \varrho$. The binary relation ϱ is called:

equiprojective, if $\mathrm{pr}_1\, \varrho = \mathrm{pr}_2 \varrho$;
partially reflexive, if $\Delta_{\mathrm{pr}\varrho} \subset \varrho$;
irreflexive, i $\varrho \cap \Delta_A = \emptyset$;
symmetric, if $\overset{-1}{\varrho} = \varrho$;
transitive, if $\varrho \circ \varrho \subset \varrho$;

it is called *partial quasiequivalence* if ϱ is partially reflexive and symmetric; *partial quasiorder* if ϱ is partially reflexive and transitive; *partial equivalence* if ϱ is a symmetric partial quasiorder.

Proposition. *The following* 12 *properties of a restrictive bisemigroup are equivalent:*

1) *it satisfies the identity*

$$x \rhd y = y \lhd x\,. \tag{30}$$

It is isomorphic with a restrictive bisemigroup of

2) *symmetric irreflexive one-to-one partial transformations;*
3) *symmetric irreflexive binary relations;*
4) *symmetric binary relations;*

5) *symmetric one-to-one partial transformations;*
6) *partial equivalence relations;*
7) *partial quasiorder relations;*
8) *partial quasiequivalence relations;*
9) *partially reflexive binary relations;*
10) *equiprojective one-to-one partial transformations;*
11) *equiprojective binary relations*;
12) *antisymmetric partial quasiorder relations.*

Remark. A binary relation ϱ is *antisymmetric* if $\varrho \cap \overset{-1}{\varrho} \subset \Delta_A$. Antisymmetric partial quasiorder relations may be called partial order relations, but the term „partial order relation" in the current mathematical language is used in another sense (synonymous with „order relation").

Proof. Let $(A, \rhd, \lhd)$ be a restrictive bisemigroup satisfying (30) and let $a^{-1} = a$ for every $a \in A$. Then $(A, \rhd, \lhd, {}^{-1})$ turns out to be an involuted restrictive bisemigroup satisfying (8). Let P be the isomorphism of $(A, \rhd, \lhd, {}^{-1})$ constructed in the proof of the Theorem of representation. For every $a \in A$, $P(a)$ is a one-to-one partial transformation. Clearly, $P(a)$ is a symmetric binary relation. Let for some $c \in C$ $(c, c) \in P(a)$. Then $(c, c) \in \varphi_b$ for some $b \in A$, $(b, a) \in \alpha$. This means that $R(b) \cap R(e) \neq \emptyset$, where e is the identity of the group $F(A)$. By simple transitivity of R, $b = e$, whence $e \in A$. But this is impossible, hence $P(a)$ is irreflexive.

We proved that 1) $\to$ 2). The following implications are trivial:

2) $\to$ 3) $\to$ 4) $\to$ 11), 2) $\to$ 5) $\to$ 10) $\to$ 11).

Define $Q(a) = P(a) \cup \Delta_{\mathrm{pr}\, P(a)}$. It is a matter of straightforward computation to check up that Q is an isomorphism of $(A, \rhd, \lhd, {}^{-1})$ onto an involuted restrictive bisemigroup of binary relations (if $Q(a) = Q(b)$, then $P(a) = Q(a) \setminus \Delta_{\mathrm{pr}\, Q(a)} = Q(b) \setminus \Delta_{\mathrm{pr}\, Q(b)} = P(b)$ and $a = b$). Binary relations $Q(a)$ are symmetric and partially reflexive. Without difficulty one can obtain $Q(a) \circ Q(a) = Q(a)$, i. e. $Q(a)$ for every $a \in A$ is a partial equivalence relation. Hence, 1) $\to$ 6).

The following implications are trivial: 6) $\to$ 7) $\to$ 9) $\to$ 11), 6) $\to$ 8) $\to$ 11) and 12) $\to$ 11).

Let $(\Phi, \rhd, \lhd)$ be a restrictive bisemigroup of equiprojective binary relations between the elements of a set B and $\varphi, \psi \in \Phi$. Then $\mathrm{pr}_2(\varphi \rhd \psi) = \mathrm{pr}_1(\varphi \rhd \psi) = \mathrm{pr}_1\varphi \cap \mathrm{pr}_1\psi = \mathrm{pr}_2\varphi \cap \mathrm{pr}_2\psi$, whence $\varphi \rhd \psi \subset (\mathrm{pr}_1\varphi \times B) \cap (B \times \mathrm{pr}_2\varphi) = \mathrm{pr}_1\varphi \times \mathrm{pr}_2\varphi$. This inclusion implies $\varphi \rhd \psi = \psi \cap (\mathrm{pr}_1\varphi \times \mathrm{pr}_2\varphi)$. In a quite analogous way we prove that $\psi \lhd \varphi = \psi \cap (\mathrm{pr}_1\varphi \times \mathrm{pr}_2\varphi) = \varphi \rhd \psi$, hence, $(\Phi, \rhd, \lhd)$ satisfies (30) and 11) $\to$ 2).

Let us prove now that 2) $\to$ 12). Let P be any isomorphism of a restrictive bisemigroup $(A, \triangleright, \triangleleft)$ onto a restrictive bisemigroup of binary relations between the elements of sets B and C. Without any loss of generality we may suppose that $C \cap B = \emptyset$. Let $D = B \cup C$ and $Q(a) = P(a) \cup \Delta_{\mathrm{pr}\,P(a)}$. One can easily verify that $Q(a) = Q(b) \to a = b$. Let $(A, \triangleright, \triangleleft)$ satisfy (30). Then $Q(a) \triangleright Q(b) = (P(a) \triangleright P(b)) \cup (\Delta_{\mathrm{pr}\,P(a)} \triangleright P(b)) \cup (P(a) \triangleright \Delta_{\mathrm{pr}\,P(b)}) \cup$ $\cup\, (\Delta_{\mathrm{pr}\,P(a)} \triangleright \Delta_{\mathrm{pr}\,P(b)}) = P(a \triangleright b) \cup (P(a) \triangleright P(b)) \cup (\Delta_{\mathrm{pr}_1 P(a)} \triangleright \Delta_{\mathrm{pr}\,P(b)}) \cup$ $\cup\, \Delta_{\mathrm{pr}\,P(a) \cap \mathrm{pr}P(b)} = P(a \triangleright b) \cup \Delta_{\mathrm{pr}\,P(a) \cap \mathrm{pr}\,P(b)}$. Hence $\mathrm{pr}P(a) \cap \mathrm{pr}P(b) =$ $= (\mathrm{pr}_1 P(a) \cap \mathrm{pr}_1 P(b)) \cup (\mathrm{pr}_2 P(a) \cap \mathrm{pr}_2 P(b)) = \mathrm{pr}_1(P(a) \triangleright P(b)) \cup \mathrm{pr}_2(P(b) \triangleleft$ $\triangleleft\, P(a)) = \mathrm{pr}_1 P(a \triangleright b) \cup \mathrm{pr}_2 P(a \triangleright b) = \mathrm{pr}P(a \triangleright b)$, whence $Q(a) \triangleright Q(b) =$ $= Q(a \triangleright b)$. Quite analogously, $Q(a) \triangleleft Q(b) = Q(a \triangleleft b)$. Clearly, for every $a \in A$, $Q(a)$ is an antisymmetric partial quasiorder relation.

This completes the proof of our Proposition.

The representation P considered at the end of the proof yields

Corollary 1. *Every restrictive bisemigroup is isomorphic with a restrictive bisemigroup of irreflexive transitive binary relations.*

Corollary 2. *Every left restrictive semigroup is isomorphic to a left restrictive semigroup of binary relations of any type mentioned in the Proposition.*

Proof. Let $(A, \triangleright)$ be a left restrictive semigroup. Define $a \triangleleft b = b \triangleright a$ for every $a, b \in A$. Then $(A, \triangleright, \triangleleft)$ is a restrictive bisemigroup satisfying the first condition of the Proposition, whence the corollary follows.

REFERENCES

[1] Вагнер В. В., *Рестриктивные полугруппы*, Изв. высш. учебн. заведений, Математика, *31* (1962), 19—27.

[2] Шайн Б. М., *Рестриктивные биполугруппы*, Изв. высш. учебн. заведений, Математика, *44* (1965), 168—179.

[3] Вагнер В. В., *Полугруппы, ассоциированные с обобщённой грудой*, Матем. сб. (Н. С.), *52* (1960), 597—628.

[4] Шайн Б. М., *Рестриктивные биполугруппы отображений*, Изв. высш. учебн. заведений, Математика, *56* (1967), 115—121.

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