Matematický časopis

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Matematický časopis, Vol. 19 (1969), No. 4, 292--298

Persistent URL: http://dml.cz/dmlcz/126655

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SOME GEOMETRICAL EXAMPLES OF AN IMC-QUASIGROUP

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A quasigroup $Q(\cdot)$, i. e. a non-vacuous set Q with a binary operation for which each the equations $a . x = b$ and $y . a = b$ have a unique solution for any $a, b \in Q$ is called

(a) *medial if*

(1) $$(x . y) . (u . v) = (x . u) . (y . v) \text{ for any } x, y, u, v \in Q,$$

(b) *idempotent* if

(2) $$x^2 = x . x = x \text{ for any } x \in Q ,$$

(c) *commutative* if

(3) $$x . y = y . x \text{ for any } x, y \in Q .$$

A medial, idempotent and commutative quasigroup is said to be an *A-structure.*

Example 1. An n-dimensional real affine space A^n with respect to the „mid-point of the couple of points" is an operation $\cdot$.

Example 2. Let A, B, C be three different copies of the space A^n and φ_{AB} an affine mapping of A onto B. Similarly define φ_{BA}, φ_{AC}, φ_{CA}, φ_{BC}, φ_{CB}, so that $\varphi_{AB} = \varphi_{BA}^{-1}$, $\varphi_{AC} = \varphi_{CA}^{-1}$, $\varphi_{BC} = \varphi_{CB}^{-1}$, $\varphi_{AB}\varphi_{BC} = \varphi_{AC}$. On the point set $Q = A \cup B \cup C$ the following binary operation is defined: if x, y belong to one of the spaces A, B, C then $x . y$ (according to Example 1) is the mid-point of the couple x, y; if x, y belong to different spaces of A, B, C (suppose $x \in A$, $y \in B$), then we define $x . y = \varphi_{AC}^{x} . \varphi_{BC}^{y}$. It is not difficult to prove that $Q(\cdot)$ forms an A-structure.

In the following we shall write xy instead of $x . y$.

In the theory of quasigroups there are introduced two important mappings L_a-left translation and R_a-right translation by the relations

$$L_a x = ax , \ R_a x = xa .$$

Because $L_a = R_a$ holds in an A-structure Q for any $a \in Q$, we shall introduce only one mapping denoted by G_a, namely

(4) $$G_a : Q \to Q : x \to ax = xa\ ;\ x \in Q\ ;$$

this mapping is termed as a homology, according to Example 1.

It is clear that the homology G_a is a permutation of the set Q; i. e. a $1 - 1$ mapping of Q onto Q.

The inverse permutation to G_a is denoted by G_a^{-1}, the identical permutation by I. In the group of all permutations of Q the set of all homologies G_a generates a group Q, which is said to be an A-group of the structure $Q(\cdot)$.

Proposition 1. *The homologies G_a, G^a_{-1} are automorphisms of the A-structure Q. There is exactly one fixed point of G_a, namely the point a.*

Proof. From (1), (2) it follows that the A-structure is distributive, i. e.

(5) $$x\ .\ yz = xy\ .\ xz$$

holds for any x, y, $z \in Q$. Hence G_a and G_a^{-1} are automorphisms. The rest of our assertion can be checked directly by calculation.

Proposition 2. *For any A-structure*

(6) $$G_{xy}z = G_x z\ .\ G_y z\ ,$$

(7) $$G_{xy}^{-1}z = G_x^{-1}z\ .\ G_y^{-1}z\ ,$$

(8) $$x\ .\ G_x^{-1}y = y\ .$$

Proof. The identity (6) is a distributive law. To prove (7) we denote $G_{xy}^{-1}z = w$, $G_y^{-1}z = u$, $G_y^{-1}z = v$. Hence $z = w\ .\ xy$, $z = xu$, $z = yv$. Thus by using these equations and identities (1), (2), we obtain $xy\ .\ uv = xu\ .\ yv = z\ .\ z = z = w\ .\ xy$, therefore $w = uv$. According to the definition $G_x y = y\ .\ G_x x = xy$. If we multiply this equality from the left by G_x^{-1} we obtain (8).

Proposition 3. *Let $x \in Q$ be such an element for which $G_a x = G_b x$ or $G_a^{-1}x = G_b^{-1}x$ holds. Then $a = b$.*

Proof. The first assertion is clear. If $G_a^{-1}x = G_b^{-1}x$, then $bx = G_b x = G_b G_a G_a^{-1} x = G_b G_a G_b^{-1} x = G_b(a\ .\ G_b^{-1}x) = (G_b a)\ .\ x$. Thus $b = G_b a = ab \Rightarrow \Rightarrow a = b$.

Proposition 4. *For any $a \in Q$ the mapping $S_a : x \to G_x^{-1}a$ is an endomorphism.*

Proof. The proof follows directly from (7). The mapping S_a is said to be a *symmetry* with respect to the element a.

Corollary 1. (Of Proposition 1). *If there exists $x \in Q$ such that $S_a x = S_b x$, then $a = b$.*

Corollary 2. *There is exactly one invariant point of the symmetry S_a, namely the point a.*

Proposition 5. *For any $x, y, z \in Q$*

$$y \,.\, S_x y = x \,, \tag{9}$$

$$S_{xy} x = y \,, \tag{10}$$

$$(S_y x)(S_z x) = S_{yz} x \,. \tag{11}$$

Proof. From $G_y^{-1} x = S_x y$ it follows that $x = G_y S_x y = y \,.\, S_x y$. Hence (9) holds. Multiplying $xy = G_x y$ from the left by G_x^{-1} we find (10) by means of (8). Identity (11) shows that the reflection $y \rightarrow S_y$ is an automorphism.

Proposition 6. *For any $x, y, z \in Q$*

$$S_y x = x \quad \Leftrightarrow \quad x = y \,, \tag{12}$$

$$S_z x = y \quad \Leftrightarrow \quad xy = z \,. \tag{13}$$

Proof. According to the definition the equation $x = S_y x$ is equivalent to the equation $x = G_x^{-1} y$, (i. e. $G_x x = y$), hence to the equation $x = y$. If we rewrite $xy = z$ in the form $G_x y = z$ and multiply this from the left by G_x^{-1} we obtain (13).

Proposition 7. *For an arbitrary $a \in Q$*

$$S_a^2 = S_a S_a = I \,. \tag{14}$$

Proof. From (13) and (3) we obtain $S_a x = y \Leftrightarrow xy = a \Leftrightarrow S_a y = x$ thus $S_a^2 x = S_a(S_a x) = S_a y = x$.

Corollary 3. *The endomorphism S_a is an automorphism.*

The subgroup of the group of all permutations on Q generated by the set of all symmetries S_a, $a \in Q$ is termed the symmetry group (of the quasigroup Q) and is denoted by $\mathscr{S}$.

Now we turn our attention to the structure of the group $\mathscr{S}$. First the notion of transfer is to be introduced. An automorphism $S_y S_x$, $x, y \in Q$ is said to be a *transfer* (with respect to the elements x, y) on Q and will be denoted by W_{xy}.

The subgroup of the group of all permutations on Q generated by the set of all transfers W_{xy}, $x, y \in Q$ is called a *transfer group* (of the quasigroup Q) and denoted by $\mathscr{W}$.

From Proposition 7 it follows immediately:

$$W_{xx} = I \,, \tag{15}$$

(16) $$W_{xy}W_{yx} = I\,, \; W_{xy} = W_{yx}^{-1}\,,$$

(17) $$W_{zy}W_{xz} = W_{xy}\,.$$

Proposition 8. *Let $a \neq b$, a, $b \in Q$. Then there is no invariant element x of the transfer W_{ab}.*

Proof. We assume that there exists $x \in Q$ such that $W_{ab}x = x$. Making use of relation (9) and denoting $y = S_b x$, we obtain $a = yS_a y = (S_b x)(S_a S_b x) = S_b x W_{ab} x = x S_b x = b$. But this contradicts the assumption $a \neq b$.

Proposition 9. *For any a, b, $c \in Q$*

(18) $$ad = bc \quad \Leftrightarrow \quad W_{ab} = W_{cd}\,.$$

Proof. Let $ad = bc$ and $x \in Q$. If we denote $y = W_{ab}x = S_b S_a x$, then from (1) and (9) we obtain $cb = ad = (x \,.\, S_a x)(y \,.\, S_d y) = (x \,.\, S_d y)(y \,.\, S_a x) = x \,.\, S_d y[(S_b S_a x)\,(S_a x)] = (x \,.\, S_d y) b$. Thus, $c = x \,.\, S_d y = x \,.\, S_d S_b S_a x$, and also $c = x S_c x$. Comparing the last two equations we find $x \,.\, S_c x = x \,.\, S_d S_b S_a x$. Hence $S_c x = S_d S_b S_a x$ or $S_c = S_d S_b S_a$; thus we obtain $W_{cd} = S_d S_c = S_d S_d S_b S_a = S_b S_a = W_{ab}$. The converse assertion should be proved easily.

Corollary 4. *Let S_a, S_b and S_c be three symmetries on Q; then there exists one and only one symmetry S_d such that $S_d = S_a S_b S_c$.*

Proof. The element d can be found from the equation $bc = ad$. The uniqueness of S_d follows from Corollary 1.

Corollary 5. *Let W_{ab} be a transfer on Q and $p \in Q$. Then there exists one and only one element $x \in Q$ such that $W_{px} = W_{ab}$.*

Proof. x is determined by the equation $pb = ax$.

Now we shall show a very important consequence concerning the structure of the group $\mathscr{W}$.

Proposition 10. *Let p be a fixed point of Q. Then the set $\mathscr{W}_p$ of the transfers W_{px}, $x \in Q$, forms an Abelian group with respect to the operation of composition. The neutral element of $\mathscr{W}_p$ is $W_{pp} = I$ and the inverse to the element W_{px} is W_{py}, where $y = S_p x$.*

Proof. For any W_{pa}, $W_{pb} \in \mathscr{W}_p$ there is

(19) $$W_{pb} \,.\, W_{pa} = W_{pc}\,, \text{ where } pc = ab\,.$$

Indeed, defining c by the equation $pc = ab$, we obtain $W_{ac} = W_{pb}$. From (18) and regarding (17) we find $W_{pb}W_{pa} = W_{ac}W_{pa} = W_{pc}$; from $ab = ba$ and (19) we obtain $W_{pb} \,.\, W_{pa} = W_{pc} = W_{pa}W_{pb}$.

If $W_{px}W_{py} = I = W_{pp}$, then from (19) it follows that $p = pp = xy$. Hence $y = S_p x$. In accordance with Corollary 5 any element W_{ab} can be written in the form of W_{px}. Hence the group $\mathscr{W}$ is not only generated by the elements W_{ab} but $\mathscr{W}$ is directly the set of these elements and it is isomorphic with the group $\mathscr{W}_p$. It is clear that without being afraid of confusion we can identify group $\mathscr{W}_p$ with the group $\mathscr{W}$.

Proposition 11. *The group $\mathscr{W}_p$ acts on Q transitively and effectively.*

Proof. The effectivity is the direct consequence of Proposition 8.

For any $x, y \in Q$ we determine c by the equation $cx = p \,.\, xy$ and according to (18) and (10) we find

$$W_{pc}x = W_{x,\,xy}x = S_{xy}S_x x = S_{xy}x = y\,.$$

It is known that the mapping

$$\omega_p : Q \to \mathscr{W}_p : x \to W_{px}$$

is a $1 - 1$ mapping from Q onto $\mathscr{W}_p$.

It is not difficult to see that the operation $\circ$ on the set $\mathscr{W}_p$ corresponds to the operation $\circ$ on Q.

$$W_{px} \circ W_{py} = W_{p,xy}\,.$$

The mapping ω_p is an isomorphism between the quasigroups $Q(\circ)$ and $\mathscr{W}_p(\circ)$.

Proposition 12. *Let $S \in \mathscr{S}$ and $p \in Q$ be fixed. Then there exists exactly one element $x \in Q$ such that one and only one of the following relations holds*

(a) $$S = W_{px}\,,$$

(b) $$S = S_p W_{px}\,.$$

If $S = S_p W_{px}$, then also $S = W_{pa} S_p$, where $a = S_p x$.

Proof. From Corollary 4 it follows that S may be written in one of the forms S_a, $S_a S_b$.

From Corollary 2 and Proposition 8 it follows that S may be written only in one of the forms S_a and $S_a S_b$.

From Proposition 1 it follows that if $S = S_a S_b$, then $S = W_{px}$, where x is determined uniquely. If $S = S_a$ then $S_a = S_a S_p S_p = W_{pa} S_p$. It is easy to show that there exists just one invariant element of the mapping $S_p S_a S_p$, namely the element $x = S_p a$. Thus, $S_p S_a S_p = S_x$ and therefore

$$W_{pa} S_p = S_a = S_p S_x S_p = S_p W_{px}\,.$$

Corollary 6. *The group $\mathscr{W}$ is a proper subgroup of the group $\mathscr{S}$. Now we shall show that the group $\mathscr{W}$ is a subgroup of the group $\mathscr{G}$.*

Proposition 13. *The set of all elements of the form $V_{ab} = G_a^{-1} G_b$ with respect to the composition is the group $\mathscr{W}$.*

Proof. It is necessary to verify that any transformation V_{ab} can be expressed as W_{xy} and any transformation W_{ab} can be expressed as V_{xy}. For proving this we shall use:

Lemma 1. *For any* $a, b \in Q$

(20) $$V_{ab} = W_{a,ab}.$$

Proof. By means of relations (4), (9), (5), (9) and (10) we obtain

$$G_b S_a x = b \,.\, S_a x = (x \,.\, S_b x) S_a x = (x \,.\, S_a x) \,.\, (S_a x \,.\, S_b x) = a \,.\, S_{ab} x = G_a S_{ab} x.$$

Thus

$$G_b S_a = G_a S_{ab}.$$

Having multiplied the last equation from the left by G_a^{-1} and from the right by S_a and considering (14) we obtain (20).

Now it is not difficult to prove Proposition 13. If the mapping V_{ab} is given, then W_{xy} is determined by (20). And vice versa if the transfer W_{ac} is given, then $V_{ab} = W_{ac}$ holds for $b = S_c a$.

From the foregoing statements it immediatelly follows that $\mathscr{S} \supset \mathscr{W}$, $\mathscr{G} \supset \mathscr{W}$.

Proposition 14. *For any* a, b

(21) $$V_{ab}^2 = V_{ab} V_{ab} = W_{ab}.$$

Proof. For any $x \in Q$

$$W_{a,ab} x = S_{ab} S_a x = [S_a(S_a x)]\,[S_b(S_a x)] = x(S_b S_a x) = S_b(S_b x \,.\, S_a x) =$$
$$= S_b S_{ab} x = W_{ab,b} x\,.$$

Thus

(22) $$W_{a,ab} = W_{ab,b}\,.$$

Hence

$$V_{ab} V_{ab} x = W_{ab,b} W_{a,ab} x = S_b S_{ab} S_{ab} S_a x = W_{ab} x.$$

Some of the problems of this paper are special cases of those introduced in [2]. The main difference is in the commutativity of the structures. For instance, the groups $\mathscr{W}$ and $\mathscr{G}$ are proper subgroups of the group of all regular transformations C_A (notation by Belousow [1]). It might be interesting to use Belousow's results in geometry.

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Received October 16, 1967.

Katedra matematiky a deskriptívnej geometrie Slovenskej vysokej školy technickej, Bratislava