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INDICATRIX OF BANACH AND A SPACE OF CONTINUOUS FUNCTIONS

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In paper [1] certain spaces of real functions with the Baire type metric are considered. In the present paper we shall establish connection with paper [1]. We shall investigate one class of continuous functions in the space $\Omega(T, S_t; t \in T)$, determined in paper [1], in connection with the indicatrix of Banach.

Let us introduce the definition of the space $\Omega\,(T, S_t; t \in T)$: Let $\emptyset \neq T \subset \subset \langle 1, \infty)$ and let $+\infty$ be an accumulation point of the set T. For each $t \in T$ let $S_t \subset E_1 = (-\infty, +\infty)$, where each of sets S_t has two elements at least. Then $\Omega\,(T, S_t; t \in T) = \mathop{\mathsf{X}}\limits_{t \in T} S_t$. Hence $\Omega\,(T, S_t; t \in T)$ is the set of real functions defined on T, where the point $f(t) \in S_t$ is the value of a function f at t. Let us define a metric for this set:

$$\varrho(f, g) = 1/\inf\{t \in T : f(t) \neq g(t)\}, \text{ if } f \neq g\,,$$

$$\varrho(f, g) = 0, \text{ if } f = g\,.$$

By investigating functions with a bounded variation S. Banach established a function $n(s, f)$, the so called indicatrix of Banach, which determines the number of intervals (the degenerated ones too), from which the set $\{t : s = f(t)\}$ consists. In the case, where the number of these intervals is not finite we put $n(s, f) = +\infty$.

Let us define a real function $n(s, f)$ on $E_1 \times \Omega\,(T, S_t; t \in T)$ as follows: If the number of components of the set $\{t : s = f(t)\}$ $(\subset \langle 1, \infty))$ is equal to a finite number a, then we put $n(s, f) = a$. In the reverse case we put $n(s, f) = +\infty$.

In the following we shall consider the space $\Omega\,(T, S_t; t \in T)$ with a special choice of sets T and S_t. We shall assume $T = \langle 1, \infty)$ and $S_t = S$ for each $t \in T$, where S is an interval. This space we shall denote by Ω. In the following we shall investigate a subspace C of all continuous functions of the space Ω.

Theorem 1. *Let C be a subspace of all continuous functions of the space Ω.*

Then the set

$$B = \{f \in C : \mathop{\forall}_{s \in S} \mathop{\overset{\infty}{\forall}}_{i=1} \mathop{\exists}_{u \geq i} \mathop{\exists}_{v > u} f(u) = s \neq f(v)\}$$

is a residual set in C.

Proof. Let $s \in S$. Let us put

$$B(s) = \{f \in C : \mathop{\overset{\infty}{\forall}}_{i=1} \mathop{\exists}_{u \geq i} \mathop{\exists}_{v > u} f(u) = s \neq f(v)\} .$$

Then evidently $B(s) = \bigcap_{i=1}^{\infty} \bigcup_{u \geq i} G_s(u)$, where $G_s(u) = \{f : \mathop{\exists}_{v > u} f(u) = s \neq f(v)\}$ $(u \geqslant 1)$.

Lemma 1. *The set $B(s)$ is dense in C.*

Proof of Lemma 1. We shall show that for each $f_0 \in C$ and ε $(0 < \varepsilon < 1)$ there exists $g \in B(s)$ such that $g \in K(f_0, \varepsilon) = \{f : \varrho(f, f_0) < \varepsilon\}$. We shall define the function g as follows: $g(t) = f_0(t)$ for $t \leqslant 2/\varepsilon$. Since $f_0(2/\varepsilon) \in S$ there exist numbers p, q such that $s, f_0(2/\varepsilon) \in \langle p, q \rangle \subset S$. Then we put $g(t) = \frac{1}{2}(p + q) + \frac{1}{2}(q - p) \sin (t + t_0)$ for $t > 2/\varepsilon$, where t_0 $(0 \leqslant t_0 < 2\pi)$ is determined by the condition $\frac{1}{2}(p + q) + \frac{1}{2}(q - p) \sin (2/\varepsilon + t_0) = f_0(2/\varepsilon)$. This will guarantee that g is a continuous function in $2/\varepsilon$. From the construction of the function g it follows directly that $g \in B(s)$ and $\varrho(g, f_0) \leqslant \varepsilon/2 < \varepsilon$.

Lemma 2. *The set $B(s)$ is a G_δ in C.*

Proof of Lemma 2. First we show that the set $G_s(u) = \{f : \mathop{\exists}_{v > u} f(u) = s \neq f(v)\}$ $(u \geqslant 1)$ is open in C. If $g \in K(f, 1/v)$, then $K(f, 1/v) \subset G_s(u)$, because for $g \in K(f, 1/v)$, $g(t) = f(t)$ $(t \in \langle 1, v \rangle)$ holds, hence $g \in G_s(u)$.

The set $B(s) = \bigcap_{i=1}^{\infty} \bigcup_{u \geq i} G_s(u)$ as an intersection of a countable family of open sets $\bigcup_{u \geq i} G_s(u)$ is a G_δ in C.

Lemma 3. *The set $B(s)$ is residual in C.*

Proof of Lemma 3. According to Theorem 8.4 (see p. 88) of the monograph [3] each F_σ set, the complement of which is dense, is a set of the first category. From Lemma 1 and Lemma 2 it follows (taking complements) that for each $s \in S$ the set $B(s)$ is residual.

Since (according to the assumption) the set S is an interval, then evidently the following lemma holds.

Lemma 4. *Let R be a countable dense subset of S, which includes* min S and max S *(if they exist). Then to each $s \in S - R$ there are $r_1, r_2 \in R$ such that $r_1 < s < r_2$.*

Let us prove the statement of Theorem 1. Let us form a set $B^* = \bigcap_{r \in R} B(r)$. As each of sets $B(r)$ $(r \in R)$ is according to Lemma 3 residual and R is countable, then B^* is residual also. Evidently it is sufficient to prove the inclusion $B^* \subset B$.

Let $f \in B^*$ and $s \in S - R$. From Lemma 4 it follows that there exist numbers $r_1, r_2 \in R$ such that $r_1 < s < r_2$. Since $f \in B(r_1)$, for an arbitrary natural number i there are the numbers u_1, v_1 $(i \leqslant u_1 < v_1)$ such that $f(u_1) = r_1 \neq \neq f(v_1)$. From the condition $f \in B(r_2)$ there follows the existence of the numbers u_2, v_2 $(v_1 \leqslant u_2 < v_2)$ such that $f(u_2) = r_2 \neq f(v_2)$. Then $f(u_1) < s < f(u_2)$ and from the properties of the continuous functions there follows the existence u $(u_1 < u < u_2)$ such that $f(u) = s$. We have shown that for each $s \in S - R$ and for each natural i there are the numbers u and v $(v = u_2)$ such that $i \leqslant u < < v$ and $f(u) = s \neq f(v)$. If $s \in R$, then the existence of the numbers u, v with the required qualities follows from the inclusion $B^* \subset B(s)$.

The Theorem is therefore completely proved.

Theorem 2. *Let the space C and the function $n(s, f)$ be given. Then the set*

$$A = \{f \in C : \forall_{s \in S} \; n(s, f) = +\infty\}$$

is residual in C.

Proof. The statement of Theorem 2 follows from Theorem 1 and from the evident inclusion $B \subset A$.

Remark. In paper [2] a theorem is proved analogical to Theorem 2, regarding the space $C(0, 1)$ of all real continuous functions defined on the interval $\langle 0, 1 \rangle$.

Lemma 5. *The space C is complete.*

Proof. As the space Ω is complete (see [1], Theorem 4), it is sufficient to show that the set C is closed in Ω. Let $f_n \to f$ $(f_n \in C, n = 1, 2, \ldots)$ and let $t_0 \in \langle 1, \infty)$ be arbitrary. If for $n \geqslant n_0$ we have $\varrho(f_n, f) < 1/t_0$, then $\inf \{t : f_n(t) \neq \neq f(t)\} > t_0$ and there exists $\delta > 0$ such that for $t \in \langle 1, t_0 + \delta)$ we have $f(t) = = f_{n_0}(t)$, i. e. the function f is continuous at every point t_0. Hence $f \in C$.

Corollary 1. *Let the space C and the function $n(s, f)$ be given. Then the set*

$$A = \{f \in C : \forall_{s \in S} \; n(s, f) = +\infty\}$$

is a set of the second category in C.

Proof. In a complete space C (Lemma 5) we have every residual set according to the well-known Baire Theorem (see [3], p. 80) a set of the second category. The set A, which is considered in the statement of the corollary, is according to Theorem 2 residual, consequently it is a set of the second category.

In the following we shall investigate the space Ω $(T, S_t; t \in T)$ under the assumption that the set T is countable and for each $t \in T$ we have $S_t = S$, where $S(\subset E_1)$ is an arbitrary set. Let us denote this space by Ω_S. The function $n(s,f)$ defined on $E_1 \times \Omega_S$ denotes evidently the number of points of the set $\{t : s = f(t)\}$.

Theorem 3. *Let the space Ω_S and the function $n(s,f)$ be given. Then the following statements are equivalent:*

(a) *the set S is countable,*

(b) *the set $A = \{f \in \Omega_S : \forall_{s \in S}\ n(s,f) = +\infty\}$ is residual in Ω_S.*

Proof. (a) → (b): Let us put

$$D = \{f \in \Omega_S : \forall_{s \in S} \forall_{i=1}^{\infty} \exists_{t \geq i}\ s = f(t)\}.$$

Let $s \in S$ and $t \in T$. Let us put $D(s,t) = \{f \in \Omega_S : s = f(t)\}$. The set $D(s,t)$ is open because if $f \in D(s,t)$, then $K(f, 1/t) \subset D(s,t)$.

Let us put $D(s) = \bigcap_{i=1}^{\infty} \bigcup_{t \geq i} D(s,t)$. The set $D(s)$ is evidently a G_δ in Ω_S. We shall show that $D(s)$ is dense in Ω_S. For $f \in \Omega_S$ and $\varepsilon > 0$ let us define $g \in \Omega_S$ as follows:

$$g(t) = f(t) \text{ for } t \leqslant 2/\varepsilon,$$
$$g(t) = s \quad \text{for } t > 2/\varepsilon.$$

Evidently $g \in D(s)$ and $g \in K(f, \varepsilon)$.

According to the above mentioned Theorem of the monograph [3] the set $D(s)$ $(s \in S)$ is residual and of such a quality is the set $D = \bigcap_{s \in S} D(s) \subset A$ too.

(b) → (a): This implication will be proved by contradiction. Let the set S be uncountable. As for an arbitrary $f \in \Omega_S$ the set $f(T)$ is countable, $A = \emptyset$ holds and the void set in the complete space Ω_S (see [1], Theorem 4) is not residual.

Corollary 2. *Let P be the space of all sequences with the Baire metric values in the set S $(\subset E_1)$. Let A be the set of all these $a = \{a_n\}_1^\infty \in P$, for which the set $\{n : a_n = s\}$ is infinite for each $s \in S$. Then the set A is residual in P if and only if the set S is countable.*

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