Ferdinand Gliviak; Ján Plesník

On the Existence of Certain Graphs with Diameter Two

Matematický časopis, Vol. 19 (1969), No. 4, 276--282

Persistent URL: http://dml.cz/dmlcz/126653

Matematický časopis 19 (1969), No. 4

ON THE EXISTENCE OF CERTAIN GRAPHS WITH DIAMETER TWO

FERDINAND GLIVJAK, JÁN PLESNÍK, Bratislava

Under a δ_2-graph we mean an undirected graph without loops and multiple edges of diameter two, without triangles. A δ_2-graph is called μ-irreducible if after removing an arbitrary vertex and edges incidental with this vertex a graph of diameter more than two arises. We use here the same basic notions as in [1] and [2].

In this paper it is proved that for every two natural numbers $p \geqslant 3$, $n \geqslant 6p$-9 there exists a μ-irreducible δ_2-graph with minimal degree p and with n vertices, except in the case of $p = 3$, $n = 11$.

In the papers [1] and [2] some properties of these graphs (as extension and reduction by one vertex) were studied. Let A be a subset of the vertex set of a graph G and v be a new vertex. Then the adding of the vertex v and the edges $\{(v, x) \mid x \in A\}$ to graph G is called a ν-extension of the graph G by the vertex v through the set A. In the reverse case, if u is a vertex of the graph G, then under the ν-reduction of the graph G by the vertex u we understand the deleting of the vertex u and all edges incidental to it.

Lemma 1. *For every natural* $p \geqslant 4$, $r \geqslant \max(3, p-2)$ *there exists a μ-irreducible δ_2-graph $G(p, r)$ with the minimal degree p and $n = p(r-2) - 1$ vertices.*

Proof. We shall construct the graph $G(p, r)$. (See Fig. 1). The vertex set of the graph $G(p, r)$ is denoted by

$$U = \{v_0, v_1, \ldots, v_p\} \cup \{u_1, u_2, \ldots, u_{p-2}\} \cup \bigcup_{i=1}^{p} A_i, \text{where } A_i = \{a_{i1}, a_{i2}, \ldots, a_{ir}\}.$$

Let us denote: $A_{ij} = A\colon - \{a_{ij}\}$, for $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, r$.

We determine the graph $G(p, r)$ by giving the neighbourhood for every vertex respectively:

$$\Omega(v_0) = \{v_1, v_2, \ldots v_p\}$$

$$\Omega(v_j) = \{v_0, u_1, u_2, \ldots, u_{p-2}\} \cup A_j, \text{ for } j = 1, 2\,.$$

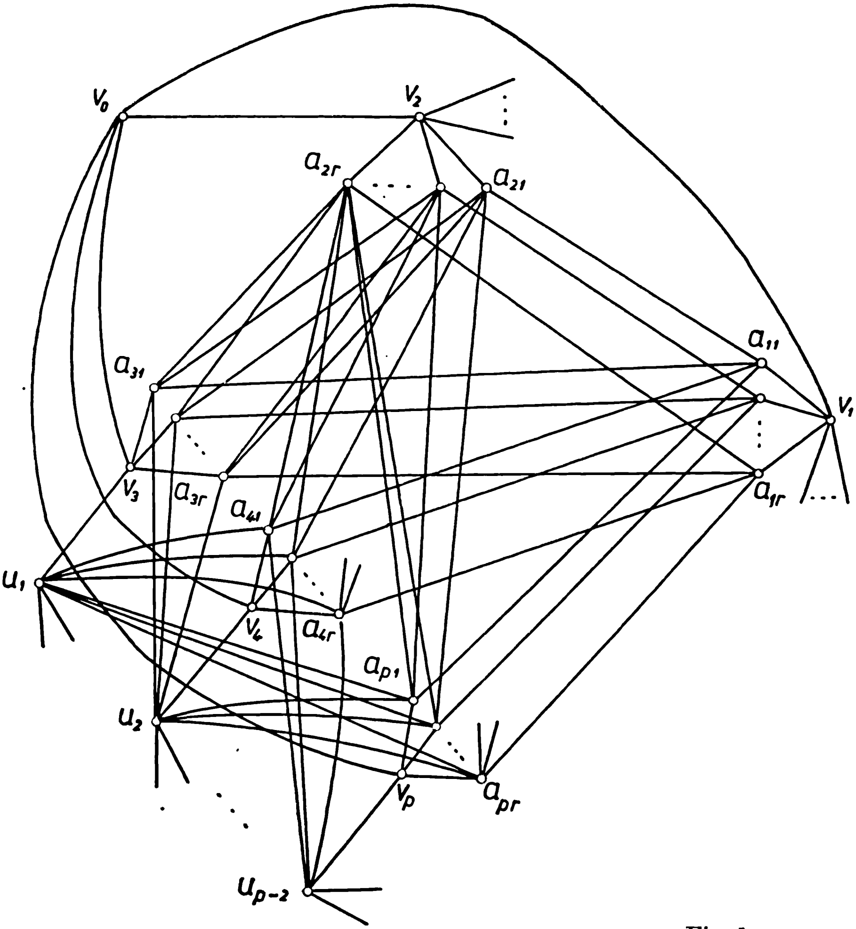

Fig. 1

$$\Omega(v_i) = \{v_0, u_{i-2}\} \cup A_i, \text{ for } i = 3, 4, \dots, p .$$

$$\Omega(u_j) = \{v_1, v_2, v_{j+2}\} \cup \bigcup_{\substack{k=1 \\ k\neq j}}^{p-2} A_k, \text{ for } j = 1, 2, \dots, p-2$$

$$\Omega(a_{1i}) = \{v_1\} \cup \bigcup_{j=2}^{p} \{a_{ji}\}, \text{ for } i = 1, 2, \dots, r$$

$$\Omega(a_{2i}) = \{v_2, a_{1i}\} \cup \bigcup_{j=3}^{p} A_{ji}, \text{ for } i = 1, 2, \dots, r$$

$$\Omega(a_{ji}) = \{v_j, a_{1i}\} \cup \bigcup_{\substack{k=1 \\ k\neq j-2}}^{p-2} \{u_k\} \cup A_{2i}, \text{ for } j = 3, 4, \dots, p;\ i = 1, 2, \dots, r .$$

It may be verified that a graph constructed according to the above construction is a δ_2-graph. The verifying is not difficult, but long and therefore it is not given. We prove that the graph $G(p, r)$ is a μ-irreducible δ_2-graph. After removing the vertex:

v_0 would be $\varrho(v_3, v_4) = 3$
v_i would be $\varrho(v_0, a_{ij}) = 3$, for $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, r$
a_{1j} would be $\varrho(v_1, a_{2j}) = 3$ and $\varrho(a_{2j}, a_{kj}) = 3$, for $j = 1, 2, \ldots, r$;
$k = 3, 4, \ldots, p$.
a_{2j} would be $\varrho(a_{1j}, a_{ik}) = 3$ and $\varrho(v_2, a_{1j}) = 3$, for $i = 3, 4, \ldots, p$;
$j = 1, 2, \ldots, r$; $k = 1, 2, \ldots, r$ and $k \neq j$;
a_{kj} would be $\varrho(v_k, a_{1j}) = 3$, for $k = 3, 4, \ldots, p$; $j = 1, 2, \ldots, r$.
u_{j-2} would be $\varrho(v_j, a_{ki}) = 3$, for $k = 3, 4, \ldots, p$; $j = 3, 4, \ldots, p$; $j \neq k$;
$i = 1, 2, \ldots, r$.

From the preceding it is clear that the graph $G(p, r)$ has $p(r + 2) - 1$ vertices. The degrees of vertices of the graph $G(p, r)$ have some of these values: p, $r + 2$, $r + p - 2$, $r + p - 1$, $(p - 2)(r - 1) + 2$, $(p - 3)r + 3$. From the conditions of Lemma 1 it follows that the minimal degree of the graph $G(p, r)$ is p. This completes the proof.

Lemma 2. *For every natural $p \geqslant 4$, $r \geqslant \max(3, p - 2)$ there exists a μ-irreducible δ_2-graph Q_k with the minimal degree p and with the number of vertices $n = p(r + 2) - 1 - k$, where $k = 1, 2, \ldots, s$ whereby $s = (p - 2)r - \max(2(p - 2), r)$.*

Proof. Let us denote $Q_0 = G(p, r)$, where $p \geqslant 4$, $r \geqslant \max(3, p - 2)$. In the graph Q_0 let us ν-reduce the vertex a_{33}. It may be seen that $\varrho(v_3, a_{13}) = 3$ and for every other pair of vertices x, y $\varrho(x, y) \leqslant 2$, By adding the edge (v_3, a_{13}) we get the δ_2-graph Q_1. This is μ-irreducible because after removing the vertex:

v_3 would be $\varrho(v_0, a_{31}) = 3$;
a_{13} would be $\varrho(v_1, a_{23}) = 3$. Other vertices cannot be reduced (see the proof of Lemma 1).

Further graphs Q_i, $i = 2, 3, \ldots, r - 2$ can be constructed from the graphs Q_{i-1} by the ν-reduction of the vertex $a_{3,i+2}$ and by adding the edge $(v_3, a_{1,i+2})$. The μ-irreducibility of graph Q_i can be verified analogically as in graph Q_1. In the set A_3 the vertices a_{31}, a_{32} remained for the sake of $\varrho(v_3, x) = 2$ for all $x \in A_2$.

A graph Q_j, $r - 2 < j \leqslant (p - 3)(r - 2)$ is constructed from the graph Q_{j-1} as follows. The graph Q_j, for $q(r - 2) < j \leqslant (q + 1)(r - 2)$, where $q = 1, 2, \ldots, p - 4$ is constructed from the graph Q_{j-1} analogically:

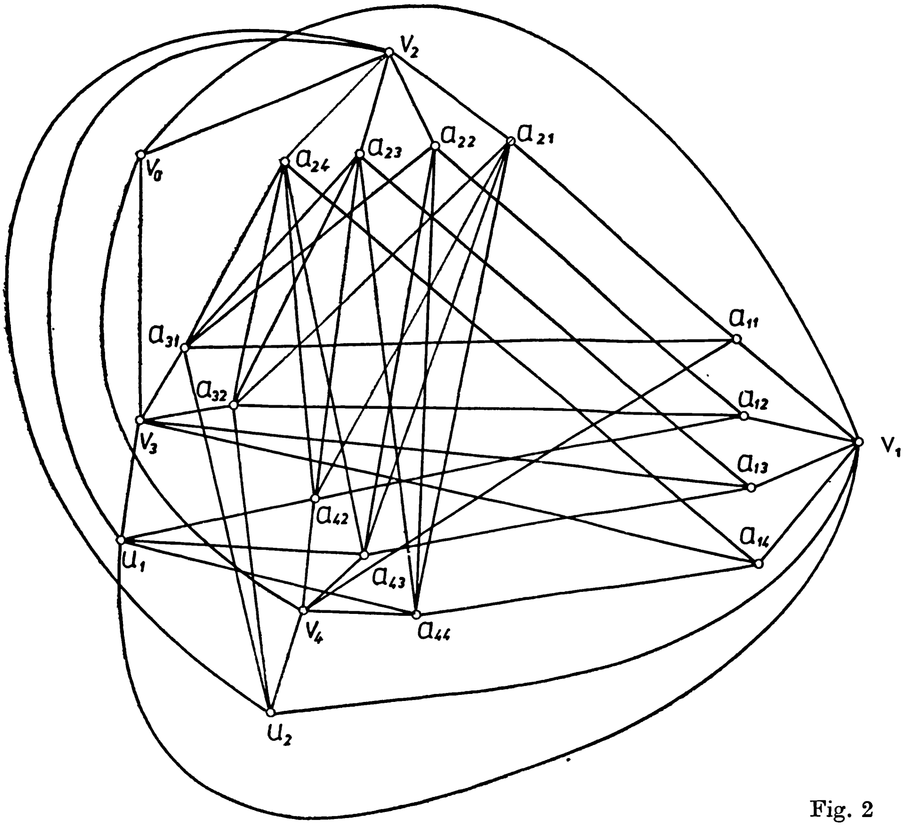

Fig. 2

1) by sequential reduction of vertices of the set A_{q+3}, except $a_{q+3,h-1}$ and $a_{q+3,h}$ where $h \equiv 2(q+1) \pmod r$.

2) after the ν-reduction of the vertex $a_{q+3,k}$ we add the edge $(v_{q+3}, a_{1,k})$.

Graphs Q_j for $(p-3)(r-2) < j \leqslant s$ can be constructed from the graphs Q_{j-1} as follows:

a) Let $2(p-2) \geqslant r$, then $s = (p-2)(r-2)$. Then we reduce sequentially all vertices from the set A_p except $a_{p,h-1}$ and $a_{p,h}$, where $h \equiv 2(p-2) \pmod r$ and after reduction of the vertex $a_{p,k}$ we add the edge $(v_p, a_{1,k})$.

b) Let $2(p-2) < r$, then $s = r(p-3)$. Then let all vertices from the set A_p be sequentially ν-reduced, except the vertices $a_{p,j}$, where $j = 2(p-2) - 1, \ldots, r$ and after the ν-reduction of the vertex $a_{p,k}$ add the edge (v_p, a_{1k}).

The proof of the μ-irreducibility of these δ_2-graphs is too long (however easy), therefore we do not give it. By the preceding construction of the graphs Q_j from the graph $G(p, r)$ the degrees of only the vertices of A_2 may decrease (at most by $(p - 2)(r - 2)$) and of the vertices $u_1, u_2, \ldots, u_{p-2}$ (at most by $(p - 3)(r - 2)$). Hence the minimal degree in every graph Q_j is p.

In Fig. 2 the graph Q_3 constructed from $G(4, 4)$ is given.

Lemma 3. *For every natural $n \geqslant 8$, $n \neq 11$ there exists a μ-irreducible δ_2-graph with n vertices and the minimal degree* 3.

Proof. Let $G(3, r)$ be the denotation for the section graph of the graph $G(p, r)$ formed by the following set of vertices:
$V = \{v_0, v_1, v_2, v_3\} \cup A_1 \cup A_2 \cup A_3$. It is obvious that $|V| = 3(r + 1) + 1$. In paper [2] it is proved that the graphs $G(3, r)$ are μ-irreducible δ_2-graphs with minimal degree 3.

For the graph $G(3, r)$ let the edge (a_{31}, a_{22}) be deleted and the edge (a_{31}, a_{12}) added. It can be verified that in such a case $\varrho(a_{22}, a_{11}) = 3$ and for every other two vertices x, y $\varrho(x, y) \leqslant 2$. A graph formed in such a way can be ν-extended by the vertex w_1 through the kernel $M_1 = \{a_{13}, a_{14}, \ldots, a_{1r}, v_3, a_{11}, a_{22}\}$. This graph, denoted by P_1, is a δ_2-graph which is also μ-irreducible because after the ν-reduction of the vertex: a_{31} would be $\varrho(_{23}, a_{11}) = 3$
a_{32} would be $\varrho(a_{21}, a_{12}) = 3$
a_{12} would be $\varrho(v_2, a_{22}) = 3$. The other vertices are obviously μ-irreducible.

Let further $r \geqslant 4$. For the graph P_1 let us delete the edge (a_{34}, a_{21}) and let us add the edge (a_{34}, a_{11}). Then $\varrho(a_{21}, a_{14}) = 3$ and for every other pair of vertices x, y $\varrho(x, y) \leqslant 2$. Let us ν-extend this graph by the vertex w_2 through the kernel $M_2 = \{a_{13}, a_{14}, \ldots, a_{1r}, v_3, a_{21}, a_{22}\}$. It can be verified that the just formed graph P_2 is a μ-irreducible δ_2-graph with minimal degree 3. Thus we have constructed the required graphs for $n = 13, 14$; $n \geqslant 16$. For $n = 8, 9, 10$ the required graphs have been constructed in paper [2] (see Fig. 4). For $n = 12, 15$ such graphs are in Fig. 3 and Fig. 4, respectively.

Remark 1. From the construction of the graphs $G(p, r)$ it can be seen that the graph $G(p, r)$ is a section graph of the graph $G(p_1, r_1)$ if $p \leqslant p_1$, $r \leqslant r_1$ holds.

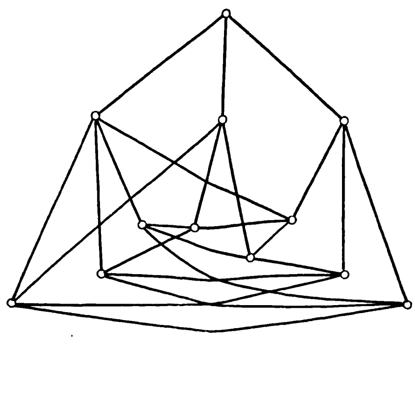

Fig. 3

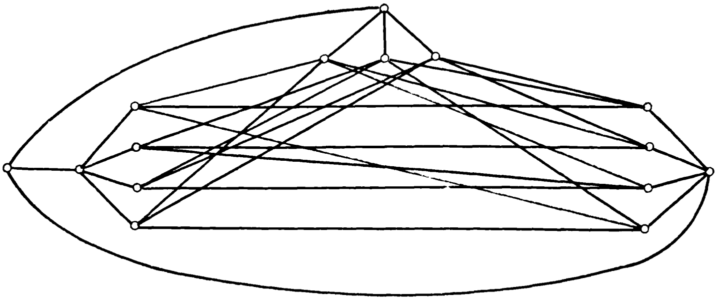

Fig. 4

Theorem 1. *For every* $p \geqslant 3$, $n \geqslant 6p - 9$ *there exists a* μ*-irreducible* δ_2*-graph with* n *vertices and the minimal degree* p, *except in the case of* $p = 3$, $n = 11$.

Remark 2. We are convinced that a μ-irreducible δ_2-graph with 11 vertices and with minimal degree 3 does not exist. We have come to this conclusion after application of Theorem 2 from paper [2] and after the investigation of every case.

Proof of Theorem 1. For $p = 3$ the Theorem is proved (Lemma 3). Let $p \geqslant 4$; let us denote $G(p, r) = (U_1, H_1)$; $G(p, r + 1) = (U_2, H_2)$. Then $|U_2| - |U_1| = p$, i. e. between the numbers of the vertices of the graphs $G(p, r)$ and $G(p, r + 1)$ there are $p - 1$ natural numbers which are not the numbers of vertices of the graph constructed by Lemma 1.

If $r > \max(3, p - 2)$ and $r \leqslant 2(p - 2)$ then in Lemma 2 we shall have $s = (p - 2)\, r - 2(p - 2) = (p - 2)(r - 2) \geqslant p - 1$. If $r > \max(3, p - 2)$ and $r > 2(p - 2)$ then in Lemma 2 we shall have $s = (p - 2)r - r = (p - 3)\, r \geqslant p - 1$. Hence in both cases the graphs Q_j, constructed from the graph $G(p, r + 1)$ by Lemma 2, fill this interval.

Thus for $n \geqslant p^2 - 1$ the required graphs exist. For $p = 4$ from the graph $G(4, 3)$ the graphs with the number of vertices 13 and 14 can be constructed by Lemma 2. For $p \geqslant 5$ the number of vertices of the graph $G(p, p - 2)$ can decrease sequentially to $p^2 - 1 - [(p - 2)^2 - 2(p - 2)] = 6p - 9$. This completes the proof.

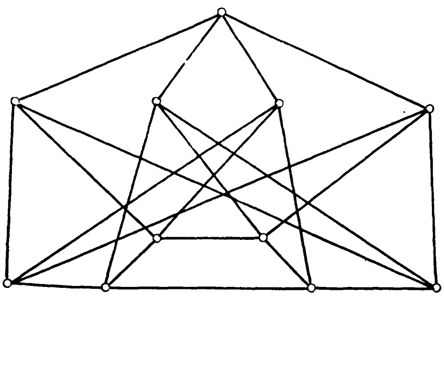

Fig. 5

Remark 3. In paper [2] it has been proved that every μ-irreducible δ_2-graph with the minimal degree 1 is the path of the length 2 and every μ-irreducible δ_2-graph with the minimal degree 2 is a 5-gon.

Corollary. *For $n = 3,5$ and for every $n \geqslant 8$ there exists a μ-irreducible δ_2-graph with the number of vertices n. For the other n such graph does not exist.*

Proof. For $n \leqslant 10$ the corollary follows from [2] (Theorem 5); for $n = 11$ the required graph is in Fig. 5; for $n > 11$ the assertion follows from Lemma 3.

REFERENCES

[1] Glivjak F., Kyš P., Plesník J., *On the extension of graphs with a given diameter without superfluous edges*, Mat. časop. *19* (1969), 92—101.

[2] Glivjak F., Kyš P., Plesník J., *On irreducible graphs with diameter two without triangles*, Mat. časop. *19* (1969), 149—157.

[3] Ore O., *Theory of graphs*, Amer. math. soc. Colloq. publ. 38 (1962).

Received September 18, 1967.

Katedra matematickej štatistiky
Prírodovedeckej fakulty
Univerzity Komenského
Bratislava