Miroslav Fiedler; Vlastimil Pták

Some Inequalities for the Spectrum of a Matrix

Matematicko-fyzikálny časopis, Vol. 10 (1960), No. 3, 148--166

Persistent URL: `http://dml.cz/dmlcz/126652`

SOME INEQUALITIES FOR THE SPECTRUM OF A MATRIX

MIROSLAV FIEDLER, VLASTIMIL PTÁK, Praha

Introduction. It is the purpose of the present paper to prove several results which enable us to associate with every matrix a region of the complex plane which contains the spectrum of the matrix considered. All known results of this type consist in formulas which use absolute values of the elements of the given matrix (see below). In distinction to these theorems, our results are based on the use of a norm of the whole non-diagonal part of the matrix. Our results are valid for a fairly wide range of norms, including especially all l_p-norms.

Further, the results of the present paper are proved for matrices partitioned into blocks and make clear the different role played by the diagonal and nondiagonal blocks.

The paper is divided into eight sections. In the first one, some auxiliary results and definitions are collected. The second and third paragraph contain sufficient conditions for the regularity of a matrix. In sections four and five, these conditions are applied to matrices $\lambda E - A$ to obtain inequalities for the proper values of A. In the sixth section we apply tensor products of linear spaces to obtain some auxiliary inequalities.

The seventh and eighth sections contain several corollaries of the main results in the most important special cases.

The starting point of all previous investigations of this type was the result of Hadamard on matrices with "dominant diagonal elements" stating that a matrix (a_{ik}) is regular if $|a_{ii}| > \sum_{k \neq i} |a_{ik}|$ for each i. Applied to the matrix $\lambda E - A$ this yields the fact that the whole spectrum of A is contained in the union of the "Gershgorin circles" $|a_{ii} - \lambda| \leqq \sum_{k \neq i} |a_{ik}|$. There is an extensive literature on questions of this type; a good bibliography may be found in the monograph of Householder [4]. As for norms of matrices and tensor product, the reader may consult [1], [2] and [3].

1. Notations and lemmas

Let X be the linear space of all vectors x with complex coordinates $x_1, \ldots, x_n$. We denote by G the set of all real functions g defined on X which fulfil the following conditions:

(1) $g(x_1 + x_2) \leqq g(x_1) + g(x_2)$ for all $x_1, x_2 \in X$;

(2) $g(\lambda x) = |\lambda| g(x)$ for all $x \in X$ and every complex number λ;

(3) $g(x) = 0$ implies $x = 0$.

The functions $g \in G$ are called norms on X. To every norm $g \in G$ there corresponds an associated norm of n-rowed square-matrices as follows: for such a matrix

$$g(A) = \sup_{\substack{x \in X \\ g(x) \leqq 1}} g(Ax).$$

It is easy to verify that this matrix norm satisfies the relations

$$g(A + B) \leqq g(A) + g(B),$$
$$g(AB) \leqq g(A)\, g(B),$$
$$g(\lambda A) = |\lambda|\, g(A)$$

for any matrices A, B and complex numbers λ.

We shall denote by N the set $\{1, 2, \ldots, n\}$. With every subset $K \subset N$ we associate a projector $P(K)$ in X transforming a vector x with coordinates x_i into the vector y with the coordinates $y_i = x_i$ for $i \in K$ and $y_j = 0$ for j non $\in K$.

Definition. Let L denote the subset of those norms $g \in G$ which fulfil the following conditions:

(L_1) If $K \subset N$, then $g(P(K)) \leqq 1$;

(L_2) If $K_1, \ldots, K_r$ is a partition of N and $P_i = P(K_i)$, $i = 1, \ldots, r$, then

$$g\left(\sum_{i=1}^{r} P_i A P_i\right) \leqq \max_i g(P_i A P_i)$$

for every matrix A;

(L_3) Let $K \subset N$, $P = P(K)$, $Q = P(N - K)$;

if A is a matrix with $PAP = 0$, then

$$g(PAQ + QAP) \leqq g(A).$$

(1,1) *Let $g \in G$ be a norm which fulfills (L_1) and the following condition:*

(L_4) *If $K \subset N$, $P = P(K)$, $Q = P(N - K)$, then $g(A) \leqq \max\{g(PAQ), g(QAP)\}$ for every matrix A satisfying $PAP = QAQ = 0$.*

Then g has the property (L_3).

Proof. Suppose that a matrix A fulfills $PAP = 0$. Let us put $B = A - QAQ$. It follows that $PBP = QBQ = 0$, $PBQ = PAQ$, $QBP = QAP$, $PAQ + QAP = B$. Assuming (L_4), we see that $g(PAQ + QAP) = g(B) \leqq \max \{g(PBQ), g(QBP)\} = \max \{g(PAQ), g(QAP)\} \leqq g(A)$; the last inequality is a consequence of (L_1). This proves (L_3).

(1,2) *If x is a vector with coordinates $x_1, \ldots, x_n$, put $g_\infty(x) = \max_i |x_i|$ and $g_{(p)}(x) = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}$ for $p \geqq 1$. Then $g_\infty \in L$ and $g_{(p)} \in L$.*

Proof. It is a well known fact that $g_\infty(A) = \max_i \sum_k |a_{ik}|$ for every matrix A. Using this expression, the conditions (L) may be verified immediately. Now let $p \geqq 1$. (L_1) and (L_2) being evident, it is sufficient according to (1,1) to prove (L_4).

Let $K \subset N$, $P = P(K)$, $Q = P(N - K)$,
and let A be a matrix with $PAP = QAQ = 0$. Is is easy to see that

$$[g_{(p)}(Py)]^p + [g_{(p)}(Qy)]^p = [g_{(p)}(y)]^p$$

for every vector $y \in X$. From this fact and from $P^2 = P$, $Q^2 = Q$ it follows that, for every vector $x \in X$,

$$\begin{aligned}
[g_{(p)}(Ax)]^p &= [g_{(p)}(PAx)]^p + [g_{(p)}(QAx)]^p = \\
&= [g_{(p)}(PA(P+Q)x)]^p + [g_{(p)}(QA(P+Q)x)]^p = \\
&= [g_{(p)}(PAQx)]^p + [g_{(p)}(QAPx)]^p \leqq \\
&\leqq [g_{(p)}(PAQ)]^p [g_{(p)}(Qx)]^p + [g_{(p)}(QAP)]^p [g_{(p)}(Px)]^p \leqq \\
&\leqq ([g_{(p)}(Qx)]^p + [g_{(p)}(Px)]^p \max \{[g_{(p)}(PAQ)]^p, [g_{(p)}(QAP)]^p\} = \\
&= [g_{(p)}(x)]^p \max \{[g_{(p)}(PAQ)]^p, [g_{(p)}(QAP)]^p\}.
\end{aligned}$$

Hence

$$g_{(p)}(Ax) \leqq g_{(p)}(x) \max \{g_{(p)}(PAQ), g_{(p)}(QAP)\},$$

so that $g_{(p)}(A) \leqq \max \{g_{(p)}(PAQ), g_{(p)}(QAP)\}$. The proof is complete.

It will be convenient to introduce some further notations and conventions. Let $K \subset N$ and $P = P(K)$ and put $Z = PX$. Let $g \in L$ and let T be a linear operator which transforms Z into itself. We intend to show that the norm of T, associated with g on Z, is equal to the norm of TP, associated with g on X. To see that, let us denote by g_1 the norm induced on Z by g. The associated norm $g_1(T)$ is equal to

$$g_1(T) = \sup_{\substack{x \in Z \\ g(x) \leqq 1}} g(Tx) = \sup_{\substack{x \in Z \\ g(x) \leqq 1}} g(TPx) \leqq \sup_{\substack{x \in X \\ g(x) \leqq 1}} g(TPx) = g(TP).$$

Conversely,

$$g(TP) = \sup_{g(x)\leqq 1} g(TPx) = \sup_{g(x)\leqq 1} g_1(TPx) \leqq$$

$$\leqq g_1(T) \sup_{g(x)\leqq 1} g_1(Px) = g_1(T)\, g(P) \leqq g_1(T).$$

Thus $g_1(T) = g(TP)$. It will lead to no misunderstanding we if agree to write $g(T)$ instead of $g_1(T)$.

Finally, if B is a matrix, we define $\hat{g}(P;B) = 0$ if PBP is singular on Z, $\hat{g}(P;B) = [g(W)]^{-1}$ if PBP is a regular operator on Z and W is its inverse operator on Z. For $P = E$ we write simply $\hat{g}(B)$ instead of $\hat{g}(E; B)$. It is easy to verify that

$$g(P; B) = \inf_{\substack{x\in PX,\\ x\neq 0}} \frac{g(Bx)}{g(x)}.$$

2. A regul:rity condition for a matrix

In this paragraph we derive a generalization of the well known Hadamard regularity condition for matrices with dominant principal diagonal.

(2,1) Theorem. *Let A be a matrix, $K_1, \ldots, K_r$ a partition of N, $P_i = P(K_i)$. Let us denote by B the matrix $B = A - \sum_{i=1}^{r} P_i AP_i$ and let g be a norm $g \in G$ fulfilling conditions (L_1) and (L_2). Let $\hat{g}(P_i; A) > g(B)$ for $i = 1, \ldots, r$. Then A is regular.*

Proof. Let us put $R = \sum_{i=1}^{r} \sqrt{g(W_i)}\, P_i$, $W = \sum_{i=1}^{r} [g(W_i)]^{-1} W_iP_i$, where W_i are operators on P_iX, inverse to P_iAP_i; the operators W_i exist since $\hat{g}(P_i; A) > 0$. According to (L_1) and (L_2), we have $g(R) \leqq \max \sqrt{g(W_i)}$ and $g(W) \leqq 1$.

Now, $RAR = R\left(\sum_{i=1}^{r} P_iAP_i + B\right) R = \sum_{i=1}^{r} g(W_i)\, P_iAP_i + RBR$. It is easy to see that the matrices $\sum_{i=1}^{r} g(W_i)\, P_iAP_i$ and W are inverse to each other. Consequently, $RAR = \left(\sum g(W_i) P_iAP_i\right)(E + WRBR)$. But $g(WRBR) \leqq g(RBR) \leqq$ $\leqq [g(R)]^2 g(B) \leqq g(B) \max_i g(W_i) = \dfrac{g(B)}{\min_i \hat{g}(P_i; A)} < 1$, so that the series $\sum_{i=0}^{\infty} H^i$ is convergent for $H = -WRBR$ to the matrix $(E + WRBR)^{-1}$. Hence RAR is regular, and so is A.

3. Another regularity condition for a matrix

The results of the present paragraph are based on some inequalites for norms of matrices. These inequalities will enable us to prove a general criterion for the regularity of a matrix.

(3,1) *Let* $K \subset N$, $P = P(K)$, $Q = P(N - K)$. *Let* A *be a matrix with* $PAP = 0$; *if* $\sigma \geqq 0$ *and* $\tau \geqq 0$, *put* $B = \sigma(PAQ + QAP) + \tau QAQ$. *Then* $g(B) \leqq \max(\sigma, \tau)\, g(A)$ for every *norm* $g \in L$.

Proof. Let us put $\xi = \min(\sigma, \tau)$. Since $A = (P + Q)\, A(P + Q) = PAQ + QAP + QAQ$, we have $B = (\sigma - \xi)(PAQ + QAP) + (\tau - \xi)\, QAQ + \xi A$. According to (L_1) and (L_3), both $g(QAQ) \leqq g(A)$ and $g(PAQ + QAP) \leqq g(A)$ are fulfilled, so that $g(B) = g[(\sigma - \xi)(PAQ + QAP) + (\tau - \xi)\, QAQ + \xi A] \leqq [(\sigma - \xi) + (\tau - \xi) + \xi]\, g(A) = \max(\sigma, \tau)\, g(A)$.

(3,2) *Let* $K_1, \ldots, K_r$ *be a partition of* N, $P_i = P(K_i)$, *and let* $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_r \geqq 0$. *Let* A *be a matrix with* $P_1AP_1 = 0$. *Then for every norm* $g \in L$

$$g\left(\sum_{i,j=1}^{r} \alpha_i \alpha_j P_i A P_j\right) \leqq \alpha_1 \alpha_2 g(A).$$

Proof. Let us put $B = \sum_{i,j=1}^{r} \alpha_i \alpha_j\, P_i A P_j$. For $r = 1$ or $\alpha_2 = 0$ we have $B = 0$ and the assertion is valid. Thus, let $\alpha_2 > 0$. We put $H = \sum_{i,j=2}^{r} \alpha_i \alpha_j\, R_i A R_j$ where $R_2 = P_1 + P_2$, $R_3 = P_3, \ldots, R_r = P_r$. It is easy to verify that $P_1HP_1 = 0$ and

$$B = \frac{\alpha_1}{\alpha_2}(P_1HQ_1 + Q_1HP_1) + Q_1HQ_1$$

where $Q_1 = P_2 + \ldots + P_r$. It follows from (3,1) that

$$g(B) \leqq \max\left(\frac{\alpha_1}{\alpha_2}, 1\right) g(H) = \frac{\alpha_1}{\alpha_2}\, g(H).$$

Now $H = DAD$ where $D = \sum_{i=2}^{r} \alpha_i R_i$, so that $g(D) \leqq \alpha_2$ according to (L_3). Hence $g(H) \leqq g^2(D)\, g(A)$ and

$$g(B) \leqq \frac{\alpha_1}{\alpha_2}\, g(H) \leqq \frac{\alpha_1}{\alpha_2}\, \alpha_2^2\, g(A) = \alpha_1 \alpha_2\, g(A).$$

This completes the proof.

(3,3) *Let* $K_1, \ldots, K_r$ *be a partition of* N, $P_i = P(K_i)$. *Let* A *be a matrix with* $P_iAP_i = 0$ for $i = 1, 2, \ldots, r$, *let* D *be a matrix with* $D =$

$= \sum_{i=1}^{r} P_i D P_i$. *Then* $g(DAD) \leqq \max_{\substack{i,j\\ i \neq j}} \{g(P_i D P_i)\ g(P_j D P_j)\}\ g(A)$ *for every norm* $g \in L$.

Proof. If $D = 0$, the assertion is true. Let $D \neq 0$, so that $P_t D P_t \neq 0$ for at least one t. Let us put $\alpha_i = g(P_i D P_i)$ and $\sigma_i = \frac{1}{\alpha_i}$ for $\alpha_i \neq 0$, $\sigma_i = 0$ for $\alpha_i = 0$. For the matrix $M = \sum_{i=1}^{r} \sigma_i P_i D P_i$, we get $g(M) \leqq 1$ by (L_2), and it is easy to verify that

$$DAD = M \left(\sum_{i,j} \alpha_i \alpha_j P_i A P_j \right) M.$$

From (3,2) it follows that

$$g(DAD) \leqq g\left(\sum_{i,j} \alpha_i \alpha_j P_i A P_j \right) \leqq \max_{\substack{i,j\\ i \neq j}} (\alpha_i \alpha_j)\ g(A).$$

which completes the proof.

(3,4) Theorem. *Let* $r \geqq 2$, *let* $K_1, \ldots, K_r$ *be a partition of* N, $P_i = P(K_i)$. *Let us denote, for a given matrix* A, $B = A - \sum_{i=1}^{r} P_i A P_i$. *Let* $g \in L$ *and suppose that*

$$\hat{g}(P_i;\ A)\ \hat{g}(P_j;\ A) > g^2(B)$$

for each pair i, j $(i, j = 1, \ldots r)$, $i \neq j$. *Then* A *is regular.*

Proof. Since $\hat{g}(P_i;\ A) > 0$, it follows that $P_i A P_i$ is regular on $P_i X$ for $i = 1, \ldots, r$. Let us denote by W_i the operator on $P_i X$, inverse to $P_i A P_i$. Put $R = \sum_{i=1}^{r} \sqrt{g(W_i)}\, P_i$, $W = \sum_{i=1}^{r} \frac{1}{g(W_i)} W_i P_i$. According to (L_1) and (L_2), we have $g(W) \leqq 1$. In the same way as in the proof of (2,1),

$$RAR = R \left(\sum_{i=1}^{r} P_i A P_i + B \right) R = \sum_{i=1}^{r} g(W_i)\, P_i A P_i + RBR =$$

$$= \left(\sum_{i=1}^{r} g(W_i)\, P_i\, A P_i \right) (E + WRBR).$$

From (2,3) we get $g(WRBR) \leqq g(RBR) \leqq$

$$\leqq \max_{i,j,i \neq j} \left(\sqrt{g(W_i)}\ \sqrt{g(W_j)} \right) g(B) = \frac{g(B)}{\min_{i,j,i \neq j} \{\hat{g}(P_i;\ A)\ \hat{g}(P_j;\ A)\}} < 1.$$

Hence RAR, as well as A, is regular.

4. The spectrum of a matrix

In this section, we shall use the criterion of regularity given in (2,1) to obtain an estimate of the spectrum of a matrix.

(4,1) *Let $K_1, \ldots, K_r$ be a partition of N, $P_i = P(K_i)$. Let A be a matrix, $B = A - \sum_{i=1}^{r} P_i A P_i$. Let g be a norm $g \in G$ which fulfills the conditions (L_1) and (L_2). Let us denote by M_i $(i = 1, 2, \ldots, r)$ the region of those complex numbers z, for which*

$$\hat{g}(P_i;\ A - zE) \leqq g(B).$$

Then, every eigenvalue of A lies at least in one M_i.

Proof. Let λ be a complex number outside every M_i. It follows that

$$\hat{g}(P_i;\ A - \lambda E) > g(B)$$

for $i = 1, 2, \ldots, r$. Consequently, the matrix $A - \lambda E$ is regular by (2,1).

5. Second theorem on the spectrum of a matrix

In this paragraph we use theorem (3,4) to obtain regions in the complex plane, containing all the eigenvalues of a given matrix.

(5,1) *Let $K_1, \ldots, K_r (r \geqq 2)$ be a partition of N, $P_t = P(K_i)$. Let A be a given matrix, $g \in L$ a norm, and let us denote by M_{ij} $(i, j = 1. \ldots, r,\ i \neq j)$ the region of those complex numbers z, for which*

$$\hat{g}(P_i;\ A - zE)\, \hat{g}(P_j;\ A - zE) \leqq g^2\Big(\sum_{\substack{k,l=1 \\ k \neq l}}^{r} P_k A P_l\Big),$$

Then every eigenvalue of A lies at least in one of the regions M_{ij} $(i, j = 1, \ldots, r;\ i \neq j)$.

Proof. Let λ be a complex number such that λ non $\in M_{ij}$ for $i, j = 1, \ldots, r$ and $i \neq j$; i. e.

$$\hat{g}(P_i; A - \lambda E)\, \hat{g}(P_i; A - \lambda E) > g^2\Big(\sum_{\substack{k,l=1 \\ k \neq l}}^{r} P_k A P_l\Big).$$

Since $\sum_{\substack{k,l=1 \\ k \neq 1}}^{r} P_k A P_l = A - \lambda E - \sum_{i=1}^{r} P_i\,(A - \lambda E)\, P_i$, it follows immediately from theorem (3,4) that $A - \lambda E$ is regular.

(5,2) Theorem. *Let $K_1, \ldots, K_r$ ($r \geqq 2$) be a partition of N, k_j the number of elements of K_j, $P_j = P(K_j)$. Let us define for $0 < \xi < 1$ the function $v(\xi) = \frac{1}{\xi}(1 - \sqrt{1 - \xi^2})$, $v(0) = 0$. Let A be a matrix, $B = A - \sum_{j=1}^{r} P_j A P_j$, $g \in L$ a norm. If i is a given index ($1 \leqq i \leqq r$), let*

$$c_i = \min_{k \neq i} \{\inf_{\lambda} [\hat{g}(P_i; A - \lambda E) + \hat{g}(P_k; A - \lambda E)]\}.$$

If $c_i > 0$ and

$$\sigma_i = \frac{2g(B)}{c_i} < 1,$$

then the region H_i of all complex numbers z such that

$$\hat{g}(P_i; A - zE) \leqq g(B)\, v(\sigma_i),$$

contains exactly k_i eigenvalues of A (each of them considered with the corresponding multiplicity).

All remaining eigenvalues are contained in the region $H = \bigcup_{\substack{j=1 \\ j \neq i}}^{r} H_j^*$ where H_j^* is the set of all complex numbers for which $\hat{g}(P_j; A - zE) \leqq g(B)$. We have $H_i \subset H_i^*$ and H_i^* is disjoint from H.

Proof. It is easy to see that the assertion is valid if $B = 0$. Hence let $B \neq 0$. Then $0 < \sigma_i < 1$ and consequently $0 < v(\sigma_i) < 1$.

We shall prove first, that H_i^* is disjoint from each H_j^* for $j \neq i$. If, on the contrary, $\lambda_0 \in H_i^* \cap H_j^* (j \neq i)$, then $c_i \leqq \hat{g}(P_i; A - \lambda_0 E) + \hat{g}(P_j; A - \lambda_0 E) \leqq \leqq 2g(B) = \sigma_i c_i < c_i$. This is a contradiction.

Further, all the regions H_j^* and the region H_i are bounded: if $z \neq 0$, $\frac{\hat{g}(P_j; A - zE)}{|z|} = \hat{g}(P_j; E - z^{-1}A) \to \hat{g}(P_j; E)$ for $|z| \to \infty$. But $\hat{g}(P_j; E) = [g(P_j)]^{-1} \geqq 1$ according to (L_1), so that

$$g(P_j; A - zE) > \frac{|z|}{2}$$

for all sufficiently large z.

Now, let $G = H_i \cup \bigcup_{\substack{j=1 \\ j \neq i}}^{r} H_j^*$. We shall prove that all eigenvalues of A are contained in G. This will follow from theorem (3,4) if we prove that for λ non $\in G$ and each pair k, l ($k, l = 1, \ldots, r$, $k \neq l$) $g(P_k; A - \lambda E)\, \hat{g}(P_l; A - \lambda E) > > g^2(B)$, since then $A - \lambda E$ is regular.

To prove this inequality, we shall distinguish two cases:

(1) λ non $\in H_k^*$ and λ non $\in H_l^*$. Then

$$\hat{g}(P_k;\ A - \lambda E) > g(B)$$

and

$$\hat{g}(P_l;\ A - \lambda E) > g(B)$$

which implies the inequality considered.

(2) $\lambda \in H_k^*$, so that $k = i$. Further,

$$\hat{g}(P_l;\ A - \lambda E) \geqq c_i - \hat{g}(P_i;\ A - \lambda E),$$

which gives

$$\hat{g}(P_i;\ A - \lambda E)\, \hat{g}(P_l;\ A - \lambda E) \geqq \xi(c_i - \xi)$$

for $\xi = \hat{g}(P_i;\ A - \lambda E)$. Since $\lambda \in H_i^*$ and λ non $\in H_i$, we have clearly $g(B)\, v(\sigma_i) < \xi \leqq g(B)$. The function $x(c_i - x)$ is increasing for $x < \frac{c_i}{2}$, and, consequently, in the interval $\langle g(B)\, v(\sigma_i),\ g(B)\rangle$. Hence

$$\xi(c_i - \xi) > g(B)\, v(\sigma_i)(c_i - g(B)\, v(\sigma_i)) =$$

$$= g(B)\, v(\sigma_i) \left[\frac{2g(B)}{\sigma_i} - g(B)\, v(\sigma_i)\right] = g^2(B)\, v(\sigma_i) \left[\frac{2}{\sigma_i} - v(\sigma_i)\right] = g^2(B).$$

This proves the desired inequality in the second case. Now, let us denote by $A(\xi)$, $0 \leqq \xi \leqq 1$, the matrix

$$A(\xi) = \sum_{j=1}^{r} P_j A P_j + \xi \sum_{\substack{j,k=1 \\ j \neq k}}^{r} P_j A P_k.$$

If we define, in a similar way as in the theorem,

$$B(\xi) = A(\xi) - \sum_{j=1}^{r} P_j A(\xi)\, P_j$$

and the numbers $c_i(\xi)$ and $\sigma_i(\xi)$, we obtain

$$B(\xi) = \xi B,\ c_i(\xi) = c_i,\ \sigma_i(\xi) = \xi \sigma_i.$$

It is easy to see that the assumptions of the preceding considerations are fulfilled for every $\xi \in \langle 0,1\rangle$ so that, for every $\xi \in \langle 0,1\rangle$, the matrix $A(\xi) - \lambda E$ is regular, if λ lies in the complement C of G. The region C separates H_i from $\bigcup_{\substack{j=1 \\ j \neq i}}^{r} H_j^*$. Since the roots of a polynomial of a given degree depend continuously on its coefficients, the matrix $A = A(1)$ has the same number of eigenvalues in H_i as the matrix $A(0)$. But the matrix $A(0) - \lambda E = \sum_{j=1}^{r} (P_jAP_j - \lambda P_j)$ is singular if and only if at least one summand $P_jAP_j - \lambda P_j$ is singular in P_jX. The summand $P_iAP_i - \lambda P_i$ is singular in P_iX for k_i numbers (each considered with its multiplicity), all of them lying in H_i. If $j \neq i$, then $P_jAP_j - \lambda P_j$ is regular in P_jX for λ non $\in H_j^*$, hence for $\lambda \in H_i$, H_j^* being disjoint from H_i. It follows that H_i contains exactly k_i eigenvalues of $A(0)$, and consequently, of A (with corresponding multiplicities). The proof is complete.

(5,3) Theorem. *Let $K_1, \ldots, K_r (r \geqq 2)$ be a partition of N, k_j the number of elements of K_j, $P_j = P(K_j)$. Let A be a matrix, $B = A - \sum_{j=1}^{r} P_jAP_j$, let $g \in L$. Let i be a given index and suppose that*

$$0 < c_i' \leqq \min_{k \neq i} \{\inf_{\lambda} (\hat{g}(P_i;\ A - \lambda E) + \hat{g}(P_k;\ A - \lambda E))\}$$

and

$$\sigma_i' = \frac{2g(B)}{c_i'} < 1.$$

Then the region H_i' of all complex numbers z such that

$$\hat{g}(P_i;\ A - zE) \leqq g(B)\, v(\sigma_i')$$

contains exactly k_i proper values of A (each considered with the corresponding multiplicity).

All remaining proper values of A are contained in the region $H = \bigcup_{j=1, j \neq i}^{r} H_j^*$ where H_j^* is the set of all complex numbers z for which $\hat{g}(P_j; A - zE) \leqq g(B)$. We have $H_i' \subset H_i^*$ and H_i^* is disjoint from H.

Proof. It follows from our assumption that $0 < c_i' \leqq c_i$, where c_i is the number defined in theorem (5,2). It follows that $1 > \sigma_i' \geqq \sigma_i$. Since v is increasing in the interval $\langle 0,1 \rangle$, we have $v(\sigma_i') \geqq v(\sigma_i)$. If H_i is the region defined in (5,2), we have the inclusions

$$H_i \subset H_i' \subset H_i^*.$$

Let us show now that $H_i^* \cap H$ is empty. Indeed, suppose that $\lambda \in H_i^* \cap H_j^*$ for some $j \neq i$. Hence

$$c_i' \leqq \hat{g}(P_i; A - \lambda E) + \hat{g}(P_j; A - \lambda E) \leqq 2g(B) = \sigma_i' c_i' < c_i'$$

which is a contradiction. According to (5,2), the region $H_i \subset H_i^*$ contains exactly k_i proper values of A (each considered with its multiplicity) and the region H contains the remaining ones. It follows that H_i' contains exactly k_i proper values. The proof is complete.

6. An application of tensor products

In this paragraph we shall recall some notions of the theory of tensor products. This theory will enable us to find a theorem similar to (5,2) but more convenient for applications.

Let Z be a given linear space. We denote by Z' the adjoint space of Z, i. e. the space of all linear functionals on Z. For $z' \in Z'$ and $z \in Z$ we denote by $\langle z, z' \rangle$ the value of the functional z' at the point z.

Let X and Y be two finite-dimensional linear spaces, $B(X, Y)$ the linear space of all bilinear functionals defined on the pair X, Y. The adjoint space to $B(X, Y)$ will be called the *tensor product of* X *and* Y and will be denoted by $X \otimes Y$. For $x \in X$ and $y \in Y$, the *tensor product* $x \otimes y$ of x and y is defined as that element of $X \otimes Y$, for which

$$\langle b, x \otimes y \rangle = b(x, y)$$

for all $b \in B(X, Y)$. It is easy to see that every element of $X \otimes Y$ can be written in the form $\sum_{i=1}^{n} x_i \otimes y_i$ where $x_i \in X$ and $y_i \in Y$ and n is the smaller of the dimensions of X and Y.

Further, let $L(X, Y)$ denote the linear space of all linear transformations of X into Y. We shall show that there is a natural isomorphism between the spaces $L(X, Y)$ and $X' \otimes Y$. In fact, it is not difficult to verify that the mapping β of $X' \otimes Y$ into $L(X, Y)$, which transforms the element $t = \sum_{i=1}^{n} x_i' \otimes y_i \in X' \otimes Y$ into the element $\beta(t) \in L(X, Y)$ such that $\beta(t)\, x = \sum_{i=1}^{n} \langle x, x_i' \rangle\, y_i$ for all $x \in X$, is an isomorphism between $X' \otimes Y$ and $L(X, Y)$.

In the sequel, we shall need the notion of the tensor product of linear transformations. Let X, Y, V and W be linear spaces. Let us define a linear mapping α of $L(X, Y) \otimes L(V, W)$ into $L(X \otimes V, Y \otimes W)$ in the following manner: if $A \in L(X, Y)$, $B \in L(V, W)$, let $\alpha(A \otimes B)$ be the element of $L(X \otimes V, Y \otimes W)$ defined by the relation $\alpha(A \otimes B)\,(x \otimes v) = Ax \otimes Bv$ fulfilled for each $x \in X$ and each $v \in V$. It is easy to see that α is onto and an isomorphism. We shall use this fact in the case $X = Y = X_1$, $V = W = X_2$, so that the transformations considered are operators in X_1, X_2 respectively. If $e_1, \ldots, e_m$ is a basis of X_1, $f_1 \ldots, f_n$ a basis of X_2, we define the matrix of the operator $A \otimes B$ in these bases as the matrix of the operator $\alpha(A \otimes B) \in L(X_1 \otimes X_2, X_1 \otimes X_2)$ in the basis $e_i \otimes f_j$. It will be denoted by $[A] \otimes [B]$ where $[A]$ and $[B]$ are matrices of A and B in the respective bases.

Finally, let us define a mapping γ of $L(X_1, X_1) \otimes L(X_2, X_2)$ into $L[L(X_1', X_2),$

$L(X_1', X_2)]$ where X_1, X_2 are linear spaces. This mapping γ will transform an element $t \in L(X_1, X_1) \otimes L(X_2, X_2)$ into the element $\gamma(t) \in L[L(X_1', X_2), L(X_1', X_2)]$ such that

$$\gamma(t)\,\xi = \beta\varkappa(t)\,\beta^{-1}\xi.$$

for each $\xi \in L(X_1', X_2)$. Here β is the isomorphic mapping of $X_1 \otimes X_2$ onto $L(X_1', X_2)$ and $\varkappa$ the isomorphic mapping of $L(X_1, X_1) \otimes L(X_2, X_2)$ onto $L(X_1 \otimes X_2, X_1 \otimes X_2)$ defined above. It is easy to see that γ is an isomorphism.

Now, let us turn to the case when normed spaces are considered. Let g and h be norms in X and Y respectively; we define a norm $p = \tau(g, h)$ in $L(X, Y)$ in the following manner. If $A \in L(X, Y)$, we put

$$p(A) = \sup\,(h(Ax);\ g(x) \leqq 1).$$

This is the usual norm of a linear transformation. If $X = Y$, we have the case of linear operators in X; it is then customary to write simply g for $\tau(g, g)$. If Y is the real line E_1, we have the case of linear functionals on X. The norm $\tau(g, |\,.\,|)$ on $L(X, E_1) = X'$ is called the adjoint norm of g and will be denoted by g'. Thus

$$g'(y') = \sup\,(|\langle x, y'\rangle|;\ \ g(x) \leqq 1).$$

If X and Y are linear spaces with norms g and h, we define a function $\hat{g} = \hat{\tau}(g, h)$ in the following manner: if $A \in L(X, Y)$, we put

$$\hat{g}(A) = \inf\,(h(Ax);\ \ g(x) \geqq 1).$$

Clearly we have $g(A) = 0$ if A is singular. If A is regular, it is easy to show that $q(A) = (p(A^{-1}))^{-1}$, where $p = \tau(h, g)$ on $L(Y, X)$. If $X = Y$, we write simply $\hat{g}$ for $\hat{\tau}(g, g)$ in conformity with the convention already introduced for matrices.

Further, it will be necessary to introduce a norm into tensor products. There are many ways of defining a reasonable norm in $X \otimes Y$. A norm ϑ in $X \otimes Y$ is said to be a crossnorm of g and h if

$$\vartheta(x \otimes y) = g(x)\,h(y)$$

for all $x \in X$ and $y \in Y$. Let ϑ be an arbitrary crossnorm of g and h and let $u \in X \otimes Y$. If $u = \sum x_i \otimes y_i$, we have

$$\vartheta(u) \leqq \sum \vartheta(x_i \otimes y_i) = \sum g(x_i)\,h(y_i).$$

Hence $\vartheta(u) \leqq \gamma(u)$ where $\gamma = \gamma\,(g, h)$ is given by

$$\gamma(u) = \inf\left(\sum g(x_i)\,h(y_i); \sum x_i \otimes y_i = u\right).$$

It is not difficult to show that γ is a crossnorm of g and h; it follows that $\gamma(g, h)$ is the greatest crossnorm of g and h. Another crossnorm may be obtained in the following manner. There is an isomorphic mapping β of $X \otimes Y$ on $L(X', Y)$. We shall define a norm $\lambda = \lambda(g, h)$ on $X \otimes Y$ by $\lambda(u) = k(\beta(u))$ where k is the norm $\tau(g', h)$ on $L(X', Y)$. Let us show that λ is a crossnorm of g and h. Indeed, we have $\lambda(x \otimes y) = k(\beta(x \otimes y) = \sup (h(\beta(x \otimes y)x'); g'(x') \leqq$ $\leqq 1) = \sup (h(\langle x, x'\rangle y); g'(x') \leqq 1) = g(x)\, h(y)$.

Let us consider now a special case where the norms in question may be easily computed. Let p be a real number $p \geqq 1$. Suppose that X and Y are spaces with bases $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$, respectively and that the norms g and h are given by

$$g(x) = \left(\sum_1^m | \xi_i |^p\right)^{\frac{1}{p}} \text{ and } h(y) = \left(\sum_1^n | \eta_i |^p\right)^{\frac{1}{p}},$$

the numbers ξ_i and η_j being coordinates of x and y in the given bases. Let $e_1' \ldots e_m'$ be the dual basis of X'. If a_{ik} is the matrix of an $A \in L(X', Y)$ in the bases $e_1' \ldots e_m'$ and $f_1, \ldots, f_n$, put $G_{(p)}(A) = \left(\sum_{i,k} | a_{ik} |^p\right)^{\frac{1}{p}}$. We have the following lemma.

(6,1) *Let p be a real number $\geqq 1$. Let X and Y be linear spaces with l_p -norms g and h. If $u \in X \otimes Y$, put*

$$n_{(p)}(u) = G_{(p)}(\beta(u))$$

where $\beta(u) \in L(X', Y)$. Then $n_{(p)}$ is a crossnorm of g and h.

Proof. Take an $x \otimes y$ and put $A = \beta(x \otimes y)$. Take an $x' \in X'$ and put $z = Ax'$. Since $z = Ax' = \langle x, x'\rangle y$ we have $\zeta_i = \eta_i \sum \xi_k \xi_k'$, the numbers $\zeta_i, \eta_i, \xi_i, \xi_i'$ being coordinates of z, y, x, x' in the given bases. It follows that $a_{ik} = \eta_i \xi_k$ whence $\left(\sum_{i,k} | a_{ik} |^p\right)^{\frac{1}{p}} = g(x)\, h(y)$. The proof is complete.

(6,2) *Let $K_i \subset N$, $P_i = P(K_i)$ and $X_i = P_i X$ where $i = 1, 2$. Let g be a norm on X and let p be a crossnorm of g_1 and g_2 where g_i are the norms induced on X_i by g. Let $A \in L(X, X)$ and let λ be a complex number. If we write simply p for $\tau(p, p)$ in $L(X_1 \otimes X_2, X_1 \otimes X_2)$, then*

$$\hat{p}(\varkappa(A_1 \otimes E_2 - E_1 \otimes A_2)) \leqq \hat{g}(P_1;\ A - \lambda E) + \hat{g}(P_2;\ A - \lambda E),$$

where $A_i = P_i A P_i$ (considered on X_i) and E_i is the identity operator on X_i.

Proof. For $i = 1,2$ there exist non-zero vectors $y_i \in X_i$ such that

$$\frac{g(A_i y_i - \lambda y_i)}{g(y_i)} = \hat{g}(P_i;\ A - \lambda E).$$

We have, by definition of $\hat{p}$

$$\hat{p}(\varkappa(A_1 \otimes E_2 - E_1 \otimes A_2)) = \inf_{\substack{t \neq 0 \\ t \in X_1 \otimes X_2}} \frac{p(\varkappa(A_1 \otimes E_2 - E_1 \otimes A_2)\, t)}{p(t)}$$

Now the last expression is majorized by the analogous quotient with $t = y_{i1} \otimes y_2$, where $y_i \in X_i$ are defined above. This quotient is, with respect to the definition of $\varkappa$, equal to

$$\frac{p(A_1 y_1 \otimes y_2 - y_1 \otimes A_2 y_2)}{p(y_1 \otimes y_2)} = \frac{p((A_1 y_1 - \lambda y_1) \otimes y_2 - y_1 \otimes (A_2 y_2 - \lambda y_2))}{p(y_1 \otimes y_2)} \leqq$$

$$\leqq \frac{p((A_1 y_1 - \lambda y_1) \otimes y_2)}{p(y_1 \otimes y_2)} + \frac{p(y_1 \otimes (A_2 y_2 - \lambda y_2))}{p(y_1 \otimes y_2)}$$

Since p is a crossnorm of g_1, and g_2, the last sum is equal to

$$\frac{g(A_1 y_1 - \lambda y_1)}{g(y_1)} + \frac{g(A_2 y_2 - \lambda y_2)}{g(y_2)} = \hat{g}(P_1;\ A - \lambda E) + \hat{g}(P_2;\ A - \lambda E)$$

and the proof is complete.

(6,3) Theorem. *Let $K_1, \ldots, K_r (r \geqq 2)$ be a partition of N, $P_j = P(K_j)$, $X_j = P_j X$, k_j the number of elements of K_j. Let A be a matrix and put $A_j = P_j A P_j$ on X_j. Let $g \in L$ and let g_j be the norms induced on X_j by g. Let p_{rs} be crossnorms of g_r and g_s and let us write simply p_{rs} for $\tau(p_{rs}, p_{rs})$ on $L(X_r \otimes X_s, X_r \otimes X_s)$. Suppose that, for some index i,*

$$c_i' = \min_{\substack{j = 1, \ldots, r \\ j \neq i}} \hat{p}_{ij}(\varkappa(A_i \otimes E_j - E_i \otimes A_j)) > 0$$

and

$$\sigma_i' = \frac{2g(B)}{c_i'} < 1$$

for $B = A - \sum_{j=1}^{r} P_j, AP_j$.

Then, the region H_i' of those complex numbers z satisfying the inequality

$$\hat{g}(P_i;\ A - zE) \leqq g(B)\, v(\sigma_i')$$

($v(x)$ defined in (5,2)) contains exactly k_i eigenvalues of the matrix A, each of them considered with the corresponding multiplicity.

All remaining eigenvalues of A are contained in the region $H = \bigcup_{\substack{j=1 \\ j \neq i}}^{r} H_j^*$ *where* H_j^* *is the region of those complex numbers z, for which*

$$\hat{g}(P_j;\ A - zE) \leqq g(B).$$

The regions H *and* H_i' *are disjoint.*

Proof. The present theorem is an immediate consequence of theorem (5,3). It is sufficient to show that the number c_i' fulfills the assumptions of (5,3); this, however, follows from (6,2).

Remark. Lemma (6,1) enables us to compute $p_{ij}(\varkappa(A_i \otimes E_j - E_i \otimes A_j))$ in the most important case when g is the l_p-norm. Then, if $\omega = \tau(G_{(p)}, G_{(p)})$ $\hat{p}_{ij}(\varkappa(A_i \otimes E_j - E_i \otimes A_j)) = \hat{\omega}([A_i] \otimes [E_j] - [E_i] \otimes [A_j])$, where $[A_i], \ldots$ are matrices of the operators $A_i, \ldots$ in the given bases. This last expression can be easily computed for $p = 1,2$ or ∞ (*see, e. g.* [3], *p.* 62—63).

7. Special cases

In this paragraph we shall spezialize some of the results obtained. First, consider the case when the sets $K_1, \ldots, K_r$ contain only one element each, so that r is equal to the order of the matrices, $r = n$.

It is easy to see that for every norm $g \in L$ and every matrix A

$$\hat{g}(P_i;\ A - \lambda E) = | a_{ii} - \lambda |,$$
$$\hat{p}[\varkappa(A_i \otimes E_j - E_i \otimes A_j)] = | a_{ii} - a_{jj} |.$$

If A is a given matrix, let $M(A)$ be the matrix with elements $m_{ii} = 0$ and $m_{ij} = a_{ij}$ for $i \neq j$.

The theorems (3,4), (5,1) and (6,2) have the following consequences:

(7,1) *Let* $A = (a_{ij})$ *be a matrix, let* $g \in L$. *Suppose that*

$$| a_{ii}a_{jj} | > g^2(M(A))$$

for all $i, j = 1, \ldots, n,\ i \neq j$. *Then A is regular.*

(7,2) *Let us denote, for a matrix* $A = (a_{ij})$ *and a norm* $g \in L$, *by* $M_{ij}(i \neq j, i, j = 1, \ldots, n)$, *the region of all complex numbers z such that*

$$| a_{ii} - z | \, | a_{jj} - z | \leqq g^2(M(A)).$$

Then, each eigenvalue of A lies at least in one of the regions M_{ij}.

(7,3) *Let* $A = (a_{ij})$ *be a matrix, let* $g \in L$. *Let* i *be a given index. Suppose that*

$$c_i = \min_{j \neq i} | a_{ii} - a_{jj} | > 0.$$

If $0 < \sigma_i = \frac{2g(M(A))}{c_i} < 1$, *then the circle*

$$| a_{ii} - z | \leq g(M(A)) \, . \, \frac{1 - \sqrt{1 - \sigma_i^2}}{\sigma_i}$$

contains exactly one eigenvalue of A.

Finally, we shall specialize the theorem (6,2) for the case when $r = 2$ and one of the sets K_i contains a single element only.

(7,4) *Let* $A = (a_{ij})$ *be a matrix. Let* $g \in L$ *and suppose that* g *fulfills* (L_4) *as well. Put*

$$\varrho = g(1, 0, \ldots, 0), \ \varrho' = g'(1, 0, \ldots, 0),$$
$$\omega = g(0, a_{21}, a_{31}, \ldots, a_{n1}), \ \omega' = g'(0, a_{12}, a_{13}, \ldots, a_{1n}).$$

Let $K = \{2, 3, \ldots, n\}$, $P = P(K)$. *Let us assume that*

$$c = \hat{g}(P; \ A - a_{11}E) > 0$$

and that

$$\sigma = \frac{2 \max (\varrho\omega', \varrho'\omega)}{c} < 1.$$

Then the circle $| a_{11} - z | \leq v(\sigma) \max (\varrho\omega', \varrho'\omega)$ *contains exactly one eigenvalue of* A. *All remaining eigenvalues of* A *are contained in the region*

$$\hat{g}(P; \ A - zE) \leq \max (\varrho\omega', \varrho'\omega),$$

which is disjoint from the above circle.

Proof. The present theorem will be an immediate consequence of (6,3) if we prove that

$$c = p(\varkappa(A_1 \otimes E_2 - E_1 \otimes A_2)),$$
$$g(B) = \max (\varrho\omega', \varrho'\omega),$$

with $A_i = P_i A P_i$ and $B = P_1 A P_2 + P_2 A P_1$ where $P_2 = P$ and $P_1 = P(K_1)$, $K_1 = \{1\}$. In the first formula, we write p for $\tau(p, p)$ where p is a crossnorm of g_1 and g_2.

Take $g(B)$ first. Put $R = P_1 A P_2$, $S = P_2 A P_1$, so that $g(B) \leq \max (g(R), g(S))$ by (L_4). According to (L_1), we have $g(R) = g(P_1 A P_2) = g(P_1 B P_2) \leq g(B)$ and similarly, $g(S) \leq g(B)$. It follows that $g(B) = \max (g(R), g(S))$. Let a_1 be the vector with coordinates $(0, a_{21}, \ldots, a_{n1})$, let a_1' be the functional with coordinates

$(0, a_{12}, \ldots, a_{1n})$. Similarly, let e_1 be the vector $(1, 0, \ldots, 0)$ and e_1' the functional $(1, 0, \ldots, 0)$. We have, for each $x \in X$

$$Rx = \langle x, a_1' \rangle \, e_1,$$
$$Sx = \langle x, e_1' \rangle \, a_1$$

whence $g(R) = g'(a_1') \, g(e_1) = \omega' \varrho$ and $g(S) = g'(e_1') \, g(a_1) = \varrho' \omega$.

Further, consider $p(\varkappa(A_1 \otimes E_2 - E_1 \otimes A_2))$. We have, the dimension of X_1 being 1,

$$p(\varkappa(A_1 \otimes E_2 - E_1 \otimes A_2)) = \inf_{\substack{t \neq 0 \\ t \in X_1 \otimes X_2}} \frac{p(\varkappa(A_1 \otimes E_2 - E_1 \otimes A_2))}{p(t)} =$$

$$= \inf_{x_1 \otimes x_2 \neq 0} \frac{p(\varkappa(A_1 \otimes E_2 - E_1 \otimes A_2)(x_1 \otimes x_2))}{p(x_1 \otimes x_2)} =$$

$$= \inf_{x_1 \otimes x_2 \neq 0} \frac{p(A_1 x_1 \otimes x_2 - x_1 \otimes A_2 x_2)}{p(x_1 \otimes x_2)} = \inf_{x_1 \otimes x_2 \neq 0} \frac{p(x_1 \otimes (a_{11} x_2 - A_2 x_2))}{p(x_1 \otimes x_2)} =$$

$$= \inf_{x_2 \neq 0} \frac{g(a_{11} x_2 - A_2 x_2)}{g(x_2)} = \hat{g}(P; a_{11} E - A).$$

8. An application to normal matrices

In this paragraph we shall specialize the preceding results in the case that the matrices considered are normal and the norm $g \in L$ is the Euclidean one. First, we shall prove two lemmas.

(8,1). *Let M_1, M_2 be two closed non-void sets of the complex plane C. Let $z \in C$.*
Then

$$\varrho(M_1, M_2) \leqq \varrho(z, M_1) + \varrho(z, M_2)$$

where ϱ denotes the distance in C.

Proof. There exist points $m_1 \in M_1$ and $m_2 \in M_2$ such that $\varrho(z, M_i) = \varrho(z, m_i)$ $(i = 1,2)$. Now $\varrho(M_1, M_2) \leqq \varrho(m_1, m_2) \leqq \varrho(z, m_1) + \varrho(z, m_2) = \varrho(z, M_1) + \varrho(z, M_2)$ which completes the proof.

(8,2). *Let A be a normal matrix, let h denote the Euclidean norm $g_{(2)}$ in X. Let z be a complex number, M the set of all eigenvalues of A.*
Then

$$\hat{h}(A - zE) = \varrho(z, M).$$

Proof. Since the matrix A is normal, there exists a unitary matrix U such that UAU^* is diagonal. According to the definition of $\hat{h}$ it is easy to see that

$$\hat{h}(A - zE) = \hat{h}(UAU^* - zE) = \min_i |\lambda_i - z|$$

where λ_i are the diagonal elements of UAU^*, consequently the eigenvalues of A. Thus, $\hat{h}(A - zE) = \varrho(z, M)$ and the proof is complete.

(8,3) Theorem. *Let A be a matrix, $K_1, \ldots, K_r$ a partition of N, $P_j = P(K_j)$, k_j the number of elements in K_j. Let the linear mappings $A_j = P_j A P_j$ be normal for $j = 1, \ldots, r$. Let $M_j (j = 1, \ldots, r)$ be the spectrum of A_j in $P_j X$, let $c_i' = \min_{j \neq i} \varrho(M_i, M_j)$ for a given index i. If $c_i' > 0$ and*

$$\sigma_i' = \frac{2h(B)}{c_i'} < 1$$

where $B = A - \sum_{j=1}^{r} P_j A P_j$,

then the spherical neighbourhood R_i of M_i consisting of those complex numbers z, fulfilling

$$\varrho(M_i, z) \leqq h(B)\, v(\sigma_i')$$

($v(x)$ was defined in (5,2) and h is the Euclidean norm), contains exactly k_i eigenvalues of A, each considered with its multiplicity. The remaining eigenvalues are contained in the region

$$\varrho(\bigcup_{j \neq i} M_j, z) \leqq h(B),$$

disjoint from the preceding one.

Proof. This is an immediate consequence of theorem (5,3) since

$$c_i' \leqq \min_{j \neq i} [\inf_z (\hat{h}(P_i; A - zE) + \hat{h}(P_j; A - zE))]$$

according to (8,1) and (8,2).

Remark. The number $\varrho(M_i, M_j)$ is equal to $\hat{\omega}([A_i] \otimes [E_j] - [E_i] \otimes [A_j])$ where $\omega = \tau(G_{(2)}, G_{(2)})$ (cf. the remark following (5,3)). This follows easily from the fact that $A_i = U_i D_i U_i^*$, $A_j = U_j D_j U_j^*$ where U_i, U_j are unitary and D_i, D_j diagonal with elements from M_i, M_j respectively.

BIBLIOGRAPHY

[1] Day M. M., *Normed linear spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete 21, Berlín 1958.

[2] Mac Duffee C. C., *The theory of matrices*, Ergebnisse der Mathematik und ihrer Grenzgebiete 5, Berlín 1933.

[3] Фаддеева В. Н., *Вычислительные методы линейной алгебры*, Москва 1950.
[4] Householder A. S., *Principles of numerical analysis*, New York 1953.

Received March 10, 1960.

Matematický ústav
Československé akademie věd v Praze

НЕКОТОРЫЕ НЕРАВЕНСТВА ДЛЯ СПЕКТРА МАТРИЦЫ

МИРОСЛАВ ФИДЛЕР и ВЛАСТИМИЛ ПТАК

Выводы

В настоящей работе рассматривается следующая задача: Пусть будет A матрица порядка n с комплексными элементами a_{ik}. Нужно определить такую область G комплексной плоскости, чтобы вес спектр матрицы A содержался в G. Результаты этого типа вытекают из исследований условий регулярности матриц. Так, например, основный результат о кругах Гершгорина вытекает из классического условия регулярности Адамара. Все известные результаты этого типа используют абсолютные величины элементов рассматриваемой матрицы. Оценки полученные в этой работе содержат только нормы недиагональной части матрицы A, при чем недиагональная часть матрицы понимается в более общем смысле, а именно так, что допускаются и матрицы разделенные в клетки.