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ON A CONSTRUCTION OF SOME SEMIGROUPS

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Dedicated to Professor Štefan SCHWARZ on the occasion of his sixtieth birthday

The purpose of this paper is to study some properties and a construction of semigroups, each left ideal of which contains a unique right identity. The main results are Theorem 6 and 8. This problem was also studied in [1] and [3]; the results are mentoined below.

In paper [2] a complete set of endomorphisms of the bicyclic semigroup is given. The present paper describes a construction of all subsemigroups of the bicyclic semigroup every left ideal of which contains a unique right identity.

Denote by S a semigroup each left ideal of which contains a unique right identity. The set of all elements which generate the left (right) principal ideal $(x)_L$ ($(x)_R$) is called the left class $L(x)$ (the right class $R(x)$). An element $e \in S$ is called a left (right) identity iff $ex = x$ ($xe = x$) for every $x \in S$. The set of all idempotents of S will be denoted by $I(S)$. The elements of $I(S)$ will be denoted by e, with indices if necessary. Further we denote e_ile_k (e_ire_k) iff $(e_i)_L \subseteqq (e_k)_L$ ($(e_i)_R \subseteqq (e_k)_R$) in S.

Remark 1. Evidently the unique right identity of $(e)_L$ is e.

Lemma 1. *For each $e_1, e_2 \in I(S)$, $e_1 \neq e_2$ there holds either e_1le_2 or e_2le_1 with $(e_1)_L \neq (e_2)_L$.*

Proof. Consider the left ideal $(e_1)_L \cup (e_2)_L$, denote its right identity by e. Then either $e \in (e_1)_L$, or $e \in (e_2)_L$. Let $e \in (e_1)_L$, hence $e = e_1$. This implies $e_2 = e_2e_1$, therefore $(e_2)_L \subseteqq (e_1)_L$, hence e_2le_1. But $(e_2)_L = (e_1)_L$ means $e_1 = e_2$, a contradiction to $e_1 \neq e_2$. Hence $(e_2)_L \subset (e_1)_L$. Similarly $e \in (e_2)_L$ implies e_1le_2.

Theorem 1. ([1], [3]). *$I(S)$ is a commutative subsemigroup of S.*

Proof. Let e_1le_2, then $e_1 = e_1e_2$. Further e_2e_1 is idempotent. Clearly $(e_2e_1)_L \subseteqq (e_1)_L$, but $(e_1)_L \subseteqq (e_2)_L$ implies $(e_1)_L \subseteqq (e_2e_1)_L$, together $(e_2e_1)_L = (e_1)_L$, where $e_1 = e_2e_1$ is the unique right identity. This togehter with $e_1 = e_1e_2$ implies $e_2e_1 = e_1e_2 = e_1$.

Lemma 2. e_1le_2 *iff* e_1re_2.

Proof. $e_1 l e_2$ implies $e_1 e_2 = e_1$. By Theorem 1 we have further $e_1 = e_2 e_1$, hence $(e_1)_R \subseteqq (e_2)_R$. This means $e_1 r e_2$. In the same way we prove that $e_1 r e_2$ implies $e_1 l e_2$.

Lemma 1 and 2 imply.

Theorem 2 ([3]). *$I(S)$ is a dually well ordered set with respect to the relation l (or r by Lemma 2). This ordering will be denoted by $\leqq$.*

Corollary 1. $e_1 \leqq e_2$ *iff* $e_1 e_2 = e_2 e_1 = e_1$.

Lemma 3. ([3]). *Each element $x \in S$ belongs to the class $L(e_1)$, where e_1 is the right identity of $(x)_L$ and to some class $R(e_2)$.*

Proof. Let e_1 be the right identity in $(x)_L$, hence $(e_1)_L \subseteqq (x)_L$. At the same time $x = xe_1$ implies $(x)_L \subseteqq (e_1)_L$, hence $(x)_L = (e_1)_L$, therefore $x \in L(e_1)$.

Further $e_1 = sx$, $s \in S$ and $x = xe_1 = xsx$, hence $(x)_R \subseteqq (xs)_R$. Now $(xs)_R \subseteqq (x)_R$, consequently $(x)_R = (xs)_R$. Since $xs = (xe_1)s = xsxs$, we have $xs = e_2 \in I(S)$ and we get $(x)_R = (e_2)_R$; this means $x \in R(e_2)$.

By Lemmas 1, 2, 3, we obtain.

Lemma 4 ([3]). *Each class $L(e)$ $(R(e))$ contains a unique idempotent e.*

Remark 2 ([3]). $L(e) \cap R(e)$ is a maximal group of S.

Remark 3. Each right ideal $(x)_R$ with $x \in R(e)$ contains a unique left identity e.

Lemma 5. *Let $x \in L(e_i)$, $e_k < e_i$. Then $xe_k \in L(e_k)$.*

Proof. Clearly $(x)_L = (e_i)_L$ implies $(xe_k)_L = (e_k)_L$.

Lemma 6. *Let $x \in L(e_i)$, $e_1 < e_2 < e_i$. Then $(xe_1)_R \subset (xe_2)_R$ and $(xe_1)_L \subset (xe_2)_L$.*

Proof. $(x)_L = (e_i)_L$ implies $e_i = sx$ for some $s \in S$. Hence $(e_1)_R \subset (e_2)_R$ implies $(xe_1)_R \subseteqq (xe_2)_R$. But $(xe_1)_R = (xe_2)_R$ implies $(sxe_1)_R = (sxe_2)_R$ for some s with $sx = e_i$. We have $(e_i e_1)_R = (e_i e_2)_R$, hence $(e_1)_R = (e_2)_R$, i.e. $e_1 = e_2$, a contradiction to $e_1 < e_2$. Hence $(xe_1)_R \subset (xe_2)_R$. Similarly by Lemma 5 we get $(xe_1)_L = (e_1)_L \subset (e_2)_L = (xe_2)_L$.

We clearly have.

Lemma 7. *Let $e_1 < e_2$. Then $(e_1 x)_L \subset (e_2 x)_L$.*

Denote $L(e_i) \cap R(e_k) = H_{ik}$.

Lemmas 6 and 7 imply.

Theorem 3. *Let $x \in H_{ik}$. Then for the chains of ideals ordered according to the inclusion we have ($\approx$ means the orderisomorphism):*

$\{(xe)_L / e \leqq e_i\} \approx \{e / e \leqq e_i\} \approx \{(xe)_R / e \leqq e_i\}$,

$\{(ex)_L / e \leqq e_k\} \approx \{e / e \leqq e_k\}$.

Denote the set $\{e / e_k \leqq e \leqq e_i\}$ by $\langle e_k, e_i \rangle$.

Denote the orderisomorphic intervals by $\langle a, b\rangle \approx \langle c, d\rangle$ (if they are finite, this means that they have the same number of elements).

Denote $L^*(e_i) = \cup \{H_{ik} \mid e_k \leqq e_i\}$.

Theorem 4. *$L^*(e_i)$ is a subsemigroup of S with the two- sided identity e_i.*

Proof. $(x)_L = (e_i)_L$ implies $(x^2)_L = (e_ix)_L = (e_ie_kx)_L = (e_kx)_L = (x)_L = (e_i)_L$ (since $x \in H_{ik}$). Similarly $(y^2)_L = (e_i)_L$. Further $(x)_L = (y)_L$ implies $(yx)_L = (x^2)_L = (e_i)_L$, $(xy)_L = (y^2)_L = (e_i)_L$. Also $(yx)_R \subset (ye_i)_R = (y)_R \subset (e_i)_R$, similarly $(xy)_R \subset (e_i)_R$. Hence $L^*(e_i)$ is a semigroup. Evidently e_i is a right identity of the semigroup $L^*(e_i)$. We further have $e_kx = x$ and $e_ix = e_i(e_kx) = (e_ie_k)x = e_kx = x$, this shows that e_i is also a left identity of $L^*(e_i)$.

Lemma 8. *Let $x \in L^*(e_i)$, $x \in R(e_k)$, $e_t < e_i$. Then $xe_t \in L^*(e_t)$, $xe_t \in R(e_s)$, where $\langle e_i, e_t\rangle \approx \langle e_k, e_s\rangle$.*

Proof. By Lemma 5 $xe_t \in L(e_t)$. By Theorem 3 there holds $\langle (xe_i)_R, (xe_t)_R\rangle \approx \langle e_i, e_t\rangle \approx \langle e_k, e_s\rangle$, where $(xe_t)_R = (e_s)_R$.

Remark 4. If $I(S) \approx \omega^*$, then $\langle e_i, e_k\rangle \approx \langle e_t, e_s\rangle$.

Theorem 5. *Let $e_k < e_i$. Then the mapping φ_k^i of $L^*(e_i)$ into $L^*(e_k)$ defined by $\varphi_k^i x = xe_k$ is a homomorphism of the semigroup $L^*(e_i)$ into $L^*(e_k)$*

Proof. By Lemma 8 we have xe_k, $ye_k \in L^*(e_k)$ for any x, $y \in L^*(e_i)$. Hence by Theorem 4 we have $(xe_k)(ye_k) \in L^*(e_k)$, $(xe_k)(ye_k) = x[e_k(ye_k)] = xye_k$.

Using Theorem 5 and Lemma 8 we get

Corollary 2. *Let $I(S) \approx \omega^*$. Let H_{ik} contains exactly one element for each $e_k \leqq e_i$. Then the samigroup $L^*(e_k)$ is isomorphic to the semigroup $L^*(e_i)$ and to $I(S)$. Here $L^*(e_i)$ is a dually well ordered set according to the inclusion of the right ideals (namely $x_2 \leqq x_1$ iff $(x_2)_R \subseteqq (x_1)_R$).*

Remark 2 and Corollary 2 imply.

Theorem 6. *If S is finite, then S is a chain of groups (namely of groups H_{ii}).*

Lemma 9. *Let $x \in H_{i1}$, $y \in H_{k2}$, $e_k \leqq e_i$. Then:*

$$xy \in H_{ks}, \text{ where } \langle e_i, e_2\rangle \approx \langle e_1, e_s\rangle.$$

Further: a) *if $e_k < e_1 < e_i$*, then $yx \in H_{j2}$, *where* $\langle e_1, e_k\rangle \approx \langle e_i, e_j\rangle$;

b) *if $e_1 \leqq e_k \leqq e_i$*, then $yx \in H_{it}$, *where* $\langle e_k, e_1\rangle \approx \langle e_2, e_t\rangle$.

Proof. By Lemma 8 we have $xy \in L^*(e_k)$. By Theorem 3 $(y)_R = (e_2)_R$ implies $(xy)_R = (xe_2)_R = (e_s)_R$, where $\langle e_i, e_2\rangle \approx \langle (xe_i)_R, (xe_2)_R\rangle \approx \langle e_1, e_s\rangle$.

a) $e_k < e_1$ implies $ye_1 = y$, hence $(x)_R = (e_1)_R$ implies $(yx)_R = (ye_1)_R = (y)_R \approx = (e_2)_R$. Further $(y)_L = (e_k)_L$ implies $(yx)_L = (e_kx)_L = (e_j)_L$. By Theorem 3 we get $\langle e_1, e_k\rangle \approx \langle (e_1x)_L, (e_kx)_L\rangle \approx \langle e_i, e_j\rangle$.

b) Since $e_1 \leqq e_k$, we get $e_kx = x$, hence $(y)_L = (e_k)_L$ implies $(yx)_L = (e_kx)_L \approx$

$= (x)_L = (e_i)_L$. Further $(x)_R = (e_1)_R$ gives $(xy)_R = (ye_1)_R = (e_t)_R$, where by Theorem 3 $\langle e_k, e_1\rangle \simeq \langle (ye_k)_R, (ye_1)_R\rangle \simeq \langle e_2, e_t\rangle$.

Lemma 10. *Let* $I(S) \approx \omega^*$. *Let* $x \in H_{i1}$, $y \in H_{i2}$, $e_2 < e_1$. *Then* $xy \in H_{ik}$, $yx \in H_{ik}$, *where* $\langle e_i, e_1\rangle \approx \langle e_2, e_k\rangle$.

As a consequence of the foregoing results we get

Theorem 7. $\mathscr{L}(e_i) = \cup \{L^*(e_k)/e_k \leqq e_i\}$ *is a subsemigroup of* S *containing the two-sieded identity* e_i.

The statements concerning $R^*(e_i) = \cup \{H_{ki}/e_k \leqq e_i\}$ will be denoted by the sign *. They can be obtained similarly as the corresponding statements for $L^*(e_i)$.

Now we consider the multiplication between the elements of $L^*(e_i)$ and $R^*(e_k)$. Using Theorems 3 and 3* it is easy to prove the following "multiplication rules":

Lemma 11. *Let* $x \in H_{ij}$, $y \in H_{tk}$.

1) *Let* $e_k \leqq e_i$. *Then* $xy \in H_{ts}$, *where* $\langle e_i, e_k\rangle \approx \langle e_j, e_s\rangle$. *Further*:

   a) *let* $e_t \leqq e_j$. *Then* $yx \in H_{sk}$, *where* $\langle e_j, e_t\rangle \approx \langle e_i, e_s\rangle$;

   b) *let* $e_j < e_t$. *Then* $yx \in H_{is}$, *where* $\langle e_t, e_j\rangle \approx \langle e_k, e_s\rangle$.

2) *Let* $e_i < e_k$, $e_t < e_j$. *Then* $xy \in H_{sj}$, *where* $\langle e_k, e_i\rangle \approx \langle e_t, e_s\rangle$ and $yx \in H_{sk}$, where $\langle e_j, e_t\rangle \approx \langle e_i, e_s\rangle$.

The foregoing results imply the validity of the following Theorem (here we use the notations: **B** is the bicyclic semigroup, L, R, H-classes of J. A Green [4]):

Theorem 8. *Let* S *be a semigroup each left ideal of which contains a unique right identity. If* $I(S) \approx \omega^*$, *then there exists a homomorphism* $f: S \to \mathbf{B}$ *with the kernel* $\ker f = H = L \cap R$ *and the image specified by the Construction C described below.*

Construction C.

Let $\mathscr{M} = \{M_\alpha/\alpha \in \Gamma\}$ be a family of sets $M_\alpha \approx \omega^*$ and let $J \approx \omega^*$ with the ordering $\leqq$.

I. [The correspondence $e_i \leftrightarrow L(e_i)$, $R(e_i)$ for the largest $e_i \in J$ for which $L(e_i) \neq \emptyset$.]

To each $e_k \in J$ we associate two elements $L(e_k)$, $R(e_k)$ of $\mathscr{M}$ in the following manner: Let e_i be the largest element in J to which we associate $L(e_i) \neq \emptyset$. Then for $e_t > e_i$ we put $R(e_t) = \emptyset$ and for $R(e_i)$ we take an arbitrary element of $\mathscr{M}$.

II. [The correspondence $(x, y) \in (L(e_i), R(e_i)) \leftrightarrow (e_k, e_j.]$
To each element $x \in L(e_i)$ we associate $\alpha_i x = e_k \leqq e_i$ and if $R(e_i) \neq \emptyset$, then to each $y \in R(e_i)$ we associate $\beta_i y = e_s \leqq e_i$, where the following contitions (1—4) are satisfied:

1) $x_1, x_2 \in L(e_i)$, $x_1 < x_2$ in $L(e_i)$ imply $\alpha_i x_1 < \alpha_i x_2$ in J.
2) For every $x_1, x_2 \in L(e_i)$ there exists an $x_3 \in L(e_i)$, where $\langle \alpha_i x_1, \alpha_i x_3 \rangle \simeq \langle e_i, \alpha_i x_2 \rangle$ and for every $y_1, y_2 \in R(e_i)$ there exists some $y_3 \in R(e_i)$, where $\langle \beta_i y_1, \beta_i y_3 \rangle \simeq \langle e_i, \beta_i y_2 \rangle$.
3) For every $x \in L(e_i)$, $\alpha_i x = e_k$ there exists some $y \in R(e_i)$ .with $\beta_i y = e_k$ and for every $y \in R(e_i)$, $\beta_i y = e_s$ there exists some $x \in L(e_i)$ with $\alpha_i x = e_s$.
4) For a fixed e_i denote the number of elements of $\langle e_i, e_k \rangle$ by $d_k + 1$. Then for every $x \in L(e_i)$, $\alpha_i x = e_k$, $y \in R(e_i)$, $\beta_i y = e_s$ there exist $x' \in L(e_i)$, $y' \in R(e_i)$ with $\alpha_i x' = \beta_i y' = e_m$, where $d_m = nd$ and d is the greatest common divisor of d_k and d_s, $n = 1, 2, 3, \ldots$.

The results of the foregoing considerations show that there is possible to choose a correspondence satisfying 1—4.

III. [The correspondence $e_k \leftrightarrow (L(e_k), R(e_k))$ for $e_k < e_i$.]
Let $e_k < e_i$. Then $L(e_k) \in \mathscr{M}$, $L(e_k) \neq \emptyset$. If $R(e_i) \neq \emptyset$, then $R(e_k) \in \mathscr{M}$, $R(e_k) \neq \emptyset$. If $R(e_i) = \emptyset$, then $R(e_k)$ is an arbitrary element of $\mathscr{M}$.

IV. [The correspondence $(x, y) \in (L(e_k), R(e_k)) \leftrightarrow (e_s, e_t)$ for $e_k < e_i$.]
To each $x \in L(e_k)$ we associate $\alpha_k x = e_s \leqq e_k$ and if $R(e_i) \neq \emptyset$, to each $y \in R(e_i)$ we associate $\beta_k y = e_t \leqq e_k$ in such a way that the above Condition 1—4 and moreover the following Conditions 5—6 are satisfied:

5) For every $x \in L(e_k)$, $y \in L(e_i)$ there exists $y' \in L(e_i)$, where $\langle \alpha_i y, \alpha_i y' \rangle \simeq \langle e_k, \alpha_k x \rangle$. Analogously for elements of $R(e_k)$ and $R(e_i)$.
6) If there exists $y \in L(e_i)$ with $\alpha_i y = e_t$, then there exists $x \in L(e_k)$, where $\langle e_i, e_t \rangle \simeq \langle e_k, \alpha_k x \rangle$.

Now, adjoin to every $L(e_i)$, $R(e_i)$ the element e_i as its greatest element.

V. [Multiplication in $L(e_i)$.]
We define the multiplication in $L(e_i)$ by the rule: $x_1 x_2 = x_2 x_1 = x_3 \in L(e_i)$, where $\langle \alpha_i x_1, \alpha_i x_3 \rangle \simeq \langle e_i, \alpha_i x_2 \rangle$.

VI. [Multiplication of couples $\in L(e_i)$, $L(e_k)$; $e_k < e_i$.]
We define the multiplication between the elements of $L(e_i)$, $L(e_k)$, $e_k < e_i$ as follows: Let $x \in L(e_i)$, $\alpha_i x = e_h$; $y \in L(e_k)$, $\alpha_k y = e_j$, then:

a) $xy \in L(e_k)$, $\alpha_k(xy) = e_t$, where $\langle e_i, e_j \rangle \simeq \langle e_h, e_t \rangle$;
b1) If $e_k \leqq e_h \leqq e_i$, then $yx \in L(e_t)$, where $\langle e_h, e_k \rangle \simeq \langle e_i, e_t \rangle$ and $\alpha_t(xy) = e_j$;
b2) If $e_h < e_k$, then $yx \in L(e_i)$, $\alpha_i(yx) = e_t$, where $\langle e_k, e_h \rangle \simeq \langle e_j, e_t \rangle$.

VII. [The "dual multiplication".]
We define the multiplication in $R(e_i)$ dually to that in $L(e_i)$.

Next we define the multiplication between the elements of $R(e_i)$ and $R(e_k)$, $e_k < e_i$ as follows: Let $x \in R(e_i)$, $\beta_i x = e_h$; $y \in R(e_k)$, $\beta_k y = e_j$; then:

a) $yx \in R(e_k)$, $\beta_k(yx) = e_t$, where $\langle e_i, e_j\rangle \approx \langle e_h, e_t\rangle$;

b1) If $e_k \leqq e_h \leqq e_i$, then $xy \in R(e_t)$, where $\langle e_i, e_t\rangle \approx \langle e_h, e_k,\rangle$ $\beta_t(xy) = e_j$;

b2) If $e_h < e_k$, then $xy \in R(e_i)$ and $\beta_i(xy) = e_t$, where $\langle e_k, e_h\rangle \approx \langle e_j, e_t\rangle$.

VIII. [Multiplication of couples $\in L(e_i)$, $R(e_k)$.]
We define the multiplication between the elements of $L(e_i)$ and $R(e_k)$ as follows: Let $x \in L(e_i)$, $\alpha_i x = e_j$; $y \in R(e_k)$, $\beta_k y = e_h$.

A. If $e_k \leqq e_i$, we define:

a) $xy \in L(e_h)$, $\alpha_h(xy) = e_s$, where $\langle e_i, e_k\rangle \approx \langle e_j, e_s\rangle$;

b1) If $e_h \leqq e_j$, then $yx \in R(e_k)$, $\beta_k(xy) = e_t$, where $\langle e_j, e_h\rangle \approx \langle e_i, e_t\rangle$;

b2) If $e_j < e_h$, then $yx \in L(e_i)$, $\alpha_i(yx) = e_t$, where $\langle e_h, e_j\rangle \approx \langle e_k, e_t\rangle$.

B. If $e_i < e_k$, we define:

a) $xy \in R(e_j)$, $\beta_j(xy) = e_s$, where $\langle e_k, e_i\rangle \approx \langle e_h, e_s\rangle$;

b1) If $e_h < e_j$, then $yx \in R(e_k)$, $\beta_k(xy) = e_t$, where $\langle e_j, e_h\rangle \approx \langle e_i, e_t\rangle$;

b2) If $e_j \leqq e_h$, then $yx \in L(e_i)$, $\alpha_i(xy) = e_s$, where $\langle e_h, e_j\rangle \approx \langle e_k, e_s\rangle$.

Remark 5. Warne [2] has described all bicyclic subsemigroups of **B**. The present construction describes (among others) a larger class of subsemigroups of **B**, namely all those subemigroups, the left ideal of which contains a unique right identity.

Of course, the class of semigroups described above is much larger as the bicyclic semigroup.

Finaly we remark that it clearly follows from the above construction that in this way we obtain all semigroups in which any left ideal contains a unique right identity.

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