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# A REMARK ON THE OSCILLATORINESS OF SOLUTIONS OF A NON-LINEAR THIRD-ORDER EQUATION

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In [2] a theorem is given (Theorem 2, p. 250) which gives sufficient conditions for a non-oscillatory solution of the equation

$$x''' + p(t)x' + q(t)x^{\alpha} = 0\,, \tag{1}$$

with $\alpha > 1$, $\alpha = m/n$, where $m$ and $n$ are nondivisible odd natural numbers, to have the properties:

$$\lim_{t\to\infty} x''(t) = \lim_{t\to\infty} x'(t) = 0, \quad \lim_{t\to\infty} |x(t)| = L \geqq 0\,.$$

It is further shown (in a Corollary) that under the hypotheses of Theorem 2 (in [2]) with the added assumption $0 < \varepsilon < q(t)$ we have for a non-oscillatory solution $x(t)$

$$\lim_{t\to\infty} x(t) = 0.$$

In the present remark it is shown that the hypotheses of Theorem 2 (in [2]) are sufficient for $L = 0$ and thus for $\lim x(t) = 0$ to hold. A further theorem is presented which gives sufficient conditions for a non-oscillatory solution $x(t)$ of (1) with $\alpha = m/n > 0$, where $m$ and $n$ are relatively prime odd natural numbers, to have the property

$$\lim_{t\to\infty} x(t) = 0$$

or

$$\liminf_{t\to\infty} |x(t)| = 0\,.$$

**Theorem 1.** *Let the hypotheses of Theorem 2 in [2] hold, i. e.: Let* $\alpha > 1$, $\alpha = m/n$, *where* $m$ *and* $n$ *are relatively prime odd natural numbers. Let the functions* $p(t)$ *and* $q(t)$ *satisfy the following conditions for sufficiently large* $t$:

1) $q(t)$ *is non-negative and continuous*:

2) $p(t)$, $p'(t)$ *are continuous and* $p(t) < 0$, $p'(t) \geqq 0$;

3) *for any constants $A$, $B$ there exists a $t_1 > t_0$ such that for all $t \geqq t_1$ we have*

$$A + Bt - \int_{t_0}^{t} Q(s)\,\mathrm{d}s < 0, \quad \textit{where} \quad Q(t) = \int_{t_0}^{t} q(s)\,\mathrm{d}s\,.$$

*Then any non-oscillatory solution $x(t)$ of the non-linear differential equation* (1) *has thc following properties for large $t$:*

a) $\operatorname{sgn} x(t) = \operatorname{sgn} x''(t) \neq \operatorname{sgn} x'(t)$, *where*

$$\operatorname{sgn} x(t) = \begin{cases} 1 & \textit{if} \quad x(t) \geqq 0 \\ -1 & \textit{if} \quad x(t) < 0; \end{cases}$$

b) $\lim_{t\to\infty} x''(t) = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x(t) = 0;$

c) $x(t), x'(t)$ *and* $x''(t)$ *are monotonous functions.*

Proof. We shall prove that $\lim_{t\to\infty} x(t) = 0$. Let $x(t)$ be any non-oscillatory solution of the differential equation (1). Thus there exists a number $t_1 \geqq t_0$ such that $x(t) \neq 0$ for all $t \geqq t$ . Since $-x(t)$ is also a solution of the differential equation (1), without loss of generality, assume that $x(t) > 0$ for all $t \geqq t_1$. Suppose that $\lim_{t\to\infty} x(t) = L > 0$. Then from (1) we have:

$$x'''(t) = -p(t)x'(t) - q(t)x^{\alpha}(t)\,;$$

now, since for sufficiently large $t$ $x'(t) < 0$, we have

$$x'''(t) \leqq -q(t)x^{\alpha}(t) < -L^{\alpha}q(t)\,.$$

Since, by assumption 3), $\lim_{t\to\infty} Q(t) = +\infty$, this leads to $x''(t) \to -\infty$ for $t \to \infty$, which is a contradiction. Thus necessarily $L = 0$.

**Theorem 2.** *Let $\alpha = m/n > 0$, where $m$ and $n$ are relatively prime odd natural numbers. Let the functions $p(t)$, $p'(t)$ and $q(t)$ be continuous and for sufficiently large $t_0$ let for all $t \geqq t_0$*

$$p(t) \geqq 0, \quad q(t) \geqq 0, \quad p'(t) \leqq 0\,.$$

*If for any constants $A$ and $B$*

$$\lim_{t\to\infty} \left(A + Bt - \int_{t_0}^{t} Q(s)\,\mathrm{d}s\right) = -\infty\,, \tag{2}$$

*where $Q(t) = \int_{t_0}^{t} q(s)\,\mathrm{d}s$, then a solution $x(t)$ of* (1), *for which*

(3) $$x''(t_0)x(t_0) - \frac{1}{2}x'^2(t_0) + \frac{1}{2}p(t_0)x^2(t_0) \leqq 0\,,$$

*is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be any non-oscillatory solution of the differential equation (1) satisfying (3). Thus there exists a number $t_1 \geqq t_0$ such that $x(t) \neq 0$ for all $t \geqq t_1$. Since $-x(t)$ is also a solution of the differential equation (1), assume without loss of generality, that $x(t) > 0$ for all $t \geqq t_1$. Then from (1) we have

(4) $$\frac{x''(t)}{x^\alpha(t)} + \frac{\alpha}{2}\frac{x'^2(t)}{x^{\alpha+1}(t)} + \int_{t_1}^{t} \frac{p(s)x'(s)}{x^\alpha(s)}\,\mathrm{d}s + \frac{\alpha(\alpha+1)}{2}\int_{t_1}^{t}\frac{x'^3(s)}{x^{\alpha+1}(s)}\,\mathrm{d}s = K_1 - \int_{t_1}^{t} q(s)\,\mathrm{d}s\,.$$

An integration from $t_1$ to $t \geqq t_1$ equality (4) gives

$$\frac{x'(t)}{x^\alpha(t)} + \int_{t_1}^{t}\frac{(t-s)p(s)x'(s)}{x^\alpha(s)}\,\mathrm{d}s + \frac{\alpha(\alpha+1)}{2}\int_{t_1}^{t}\frac{(t-s)x'^3(s)}{x^{\alpha+2}(s)}\,\mathrm{d}s \leqq$$

$$\leqq K_2 + K_1 t - \int_{t_1}^{t} Q(s)\,\mathrm{d}s\,.$$

This implies that there is no number $t_2$ such that $x'(t) \geqq 0$ holds for any $t \geqq t_2$. Thus we have two possibilities:

1) There exists a number $t_2 \geqq t_1$ such that $x'(t) \leqq 0$ for any $t \geqq t_2$.

2) For any $t_2$ there exists a number $t_3 \geqq t_2$ such that $x'(t_3) > 0$.

Now let $t_2$ be such number that for all $t \geqq t_2 \geqq t_1$ we have $K_2 + K_1 t - \int_{t_1}^{t} Q(s)\,\mathrm{d}s < 0$. We shall prove that then we have $x'(t) \leqq 0$ for any $t \geqq t_2$, i. e. the possibility 2) does not hold. Let $t_3 \geqq t_2$ be such number that $x'(t_3) > 0$ and let $x'(t_4) = 0$ for any $t_4 \geqq t_1$, $t_4 < t_3$.

Then from (1) we have:

$$x''(t)x(t) - \frac{1}{2}x'^2(t) + \frac{1}{2}p(t)x^2(t) + \int_{t_0}^{t} q(s)x^{\alpha+1}(s)\,\mathrm{d}s =$$

$$= x''(t_0)x(t_0) - \frac{1}{2}x'^2(t_0) + \frac{1}{2}p(t_0)x^2(t_0) + \frac{1}{2}\int_{t_0}^{t} p'(s)x^2(s)\,ds\,,$$

thus for all $t \geqq t_0$

$$x''(t)x(t) - x'^2(t) \leqq x''(t)x(t) - \frac{1}{2}x'^2(t) \leqq 0$$

and therefore for all $t \geqq t_1$

$$\frac{d}{dt}\left[\frac{x'(t)}{x(t)}\right] \leqq 0\,.$$

An integration from $t_4$ to $t_3$ gives

$$\frac{x'(t_3)}{x(t_3)} \leqq \frac{x'(t_4)}{x(t_4)} = 0\,,$$

which is impossible, because $x'(t_3) > 0$. Hence $x'(t) \leqq 0$ for all $t \geqq t_2$. Thus $x(t)$ is a non-increasing function with a finite lower bound so that $\lim_{t\to\infty} x(t) = L \geqq 0$.

Now suppose that $\lim_{t\to\infty} x(t) = L > 0$. Then (1) yields $x''(t) = x''(t_2) + p(t_2)x(t_2) - p(t)x(t) + \int_{t_2}^{t} p'(s)x(s)\,ds - \int_{t_2}^{t} q(s)x^{\alpha}(s)\,ds\,,$ where $t \geqq t_2$. Therefore

$$x''(t) \leqq K_3 - L^{\alpha}\int_{t_2}^{t} q(s)\,ds$$

and from this it follows that $x''(t) \to -\infty$ for $t \to \infty$, which contradicts the assumption that $x(t) > 0$ for $t \geqq t_2$.

**Theorem 3.** *Let $\alpha = m/n > 0$, where $m$ and $n$ are relatively prime odd natural numbers. Let the functions $p(t)$, $p'(t)$, $q(t)$ and $f(t)$ be continuous and for sufficiently large $t_0$ let for all $t \geqq t_0$*

$$p(t) \geqq 0, \quad q(t) \geqq 0, \quad p'(t) + |f(t)| \leqq 0\,.$$

*Suppose that* (2) *holds and that $x(t)$ is a solution of the equation*

$$x''' + p(t)x' + q(t)x^{\alpha} = f(t)\,, \tag{5}$$

*for which*

$$x''(t_0)x(t_0) - \frac{1}{2}x'^2(t_0) + \frac{1}{2}p(t_0)x^2(t_0) + \frac{1}{2}\int_{t_0}^{\infty}|f(t)|\,dt \leqq 0\,. \tag{6}$$

*Then* $x(t)$ *is either oscillatory or* $\liminf_{t\to\infty}|x(t)| = 0$.

Proof. Let $x(t) > 0$ for all $t \geqq t_1 \geqq t_0$, let $x(t)$ satisfy (6) and let $\liminf_{t\to\infty} x(t) = L > 0$. Thus there exists a number $t_1^* \geqq t_1$ such that $x(t) \geqq L_1 = L/2$ for all $t \geqq t_1^*$. From (5) we have for $t \geqq t_1^* \geqq t_1$

$$\frac{x''(t)}{x^\alpha(t)} + \int_{t^*_1}^{t}\frac{p(s)x'(s)}{x^\alpha(s)}\,ds + \frac{\alpha(\alpha+1)}{2}\int_{t^*_1}^{t}\frac{x'^3(s)}{x^{\alpha+2}(s)}\,ds \leqq K_1 - \tag{7}$$

$$-\int_{t^*_1}^{t} q(s)\,ds + \frac{1}{L_1^\alpha}\int_{t^*_1}^{t}|f(s)|\,ds$$

which, analogously as in the proof of Theorem 2, implies the existence of $t_2 \geqq t_1^*$ such that for all $t \geqq t_2$ $x'(t) \leqq 0$; thus $\lim_{t\to\infty} x(t) = L$.

Using (5), we have for $t \geqq t_2$

$$x''(t) \leqq K_3 - L^\alpha \int_{t_2}^{t} q(s)\,ds + \int_{t_2}^{t}|f(s)|\,ds$$

and using (2), we see that $x''(t) \to -\infty$ for $t \to \infty$, which contradicts the assumption that $x(t) > 0$ for all $t \geqq t_2$. Therefore $\liminf_{t\to\infty} x(t) = 0$.

Now let $x(t) < 0$ for all $t \geqq t_1 \geqq t_0$, let $x(t)$ satisfy (6) and let $\liminf_{t\to\infty}|x(t)| = L > 0$. Integrating (7) from $t_1^*$ to $t \geqq t_1^*$, we get

$$\frac{x'(t)}{x^\alpha(t)} + \int_{t^*_1}^{t}\frac{(t-s)p(s)x'(s)}{x^\alpha(s)}\,ds + \frac{\alpha(\alpha+1)}{2}\int_{t^*_1}^{t}\frac{(t-s)x'^3(s)}{x^{\alpha+2}(s)}\,ds \leqq$$

$$\leqq K_2 + K_1 t - \int_{t^*_1}^{t} Q(s)\,ds\,.$$

Since for all $t \geqq t_1^*$ $x^\alpha < 0$ holds, we have from the last inequality that there exists a number $t_2 \geqq t_1^*$ such that $x'(t) \geqq 0$ for all $t \geqq t_2$. In fact, let $x'(t_3) < 0$ and $x'(t_4) = 0$, where $t_1 \leqq t_4 < t_3$. Then from equation (5) we have:

$$x''(t)x(t) - \frac{1}{2}x'^2(t) + \frac{1}{2}p(t)x^2(t) \leqq x''(t_0)x(t_0) - \frac{1}{2}x'^2(t_0) +$$

$$+\frac{1}{2}p(t_0)x^2(t_0) + \frac{1}{2}\int_{t_0}^{t} |f(s)|\,\mathrm{d}s + \frac{1}{2}\int_{t_0}^{t} [p'(s) + |f(s)|]\,x^2(s)\,\mathrm{d}s\,,$$

and therefore

$$x''(t)x(t) - x'^2(t) \leqq x''(t)x(t) - \frac{1}{2}x'^2(t) \leqq 0$$

for all $t \geqq t_0$. If $t \geqq t_4$, then $x^2(t) \neq 0$ and

$$\frac{x'(t)}{x(t)} \leqq \frac{x'(t_4)}{x(t_4)}$$

for all $t \geqq t_4$. For $t = t_3$ we have a contradiction.

This proves the existence of $t_2 \geqq t_1^*$ such that for $t \geqq t_2$ $x'(t) \geqq 0$. Then from (5) we have

$$x''(t) \geqq K_3 + L^\alpha \int_{t_2}^{t} q(s)\,\mathrm{d}s - \int_{t_2}^{t} |f(s)|\,\mathrm{d}s$$

which, owing to (2) and (6), implies $x''(t) \to +\infty$ for $t \to \infty$ which again contradicts the assumption that $x(t) < 0$ for $t \geqq t_2$. This completes the proof.

**Theorem 4.** *Let the hypotheses be the same as in Theorem 2 with condition* (2) *replaced by*

(2′)
$$\int_{t_0}^{\infty} p(t)\,\mathrm{d}t = +\infty\,.$$

*If $x(t)$ is a solution of the equation* (1) *which satisfies the condition* (3), *then it is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that the hypotheses hold and that $x(t)$ is not oscillatory. Thus there exists a number $t_1 \geqq t_0$, such that $x(t) \neq 0$ for all $t \geqq t_1$. Then from (1) we have

$$x''(t)x(t) - \frac{1}{2}x'^2(t) + \frac{1}{2}p(t)x^2(t) \leqq x''(t_0)x(t_0) - \frac{1}{2}x'^2(t_0) +$$

$$+\frac{1}{2}p(t_0)x^2(t_0) + \frac{1}{2}\int_{t_0}^{t} p'(s)x^2(s)\,\mathrm{d}s\,,$$

thus for $t \geqq t_1$

$$x''(t)x(t) - x'^2(t) \leqq x''(t)x(t) - \frac{1}{2}x'^2(t) \leqq - \frac{1}{2}p(t)x^2(t)$$

and

$$\frac{d}{dt}\left[\frac{x'(t)}{x(t)}\right] \leqq -\frac{1}{2}p(t)\,, \tag{8}$$

and also there exists a number $t_2 \geqq t_1$ such that $x'(t)x(t) < 0$ for every $t \geqq t_2$.

Now let $x(t) > 0$ and $x'(t) < 0$. Then

$$\lim_{t\to\infty} x(t) = L \geqq 0$$

and hence $x(t) \geqq L$ for all $t \geqq t_2$. For all $t \geqq t_2$ we have

$$\frac{x'(t)}{x(t)} \geqq \frac{x'(t)}{L}$$

from which using (8) and (2′) we get $\lim_{t\to\infty} x''(t) = -\infty$, which is again contradictory to the assumption that $x(t) > 0$ for all $t \geqq t_2$.

Now let $x(t) < 0$ and $x'(t) > 0$. Then

$$\lim_{t\to\infty} x(t) = L \leqq 0\,.$$

Analogously as in the first case we prove the impossibility of $\lim_{t\to\infty} x(t) = L < 0$.

This completes the proof.

Evidently the following theorem also holds:

**Theorem 5.** *Let the hypotheses be the same as in Theorem* 3 *with condition* (2) *replaced by* (2′). *If* $x(t)$ *is a solution of the equation* (5) *which satisfies the condition* (6), *then it is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

## REFERENCES


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