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DISCONNECTING SETS AND DECOMPOSITION THEORIES

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A subset A of a topological space X is said to be *disconnecting* if $X - A$ is disconnected; A is a *minimal disconnecting subset* of X if A is a disconnecting subset of X, but A does not properly contain a disconnecting subset of X. The first three sections of this paper deal with properties of disconnecting subsets. The fourth section abstracts from the topological situation to define a *decomposition theory*; it is shown that a topology is "naturally" associated with a decomposition theory, and decomposition theory with a topology, but that a topology and its associated decomposition theory are not equivalent concepts.

I. Minimal disconnecting subsets of disconnecting sets

In certain topological spaces no disconnecting subsets contain a minimal disconnecting subset.

Example 1. Let N be the set of natural numbers with the topology containing $\emptyset$ and all complements of finite sets. Since N is connected and any infinite subset of N is homeomorphic to N, it follows that only finite subsets of N which contain at least two points are disconnected. Therefore the only disconnecting subsets are non-empty open sets which fail to contain at least two points of N. But if U is such an open set and $p \in U$, then $U - \{p\}$ is a proper subset of U which also disconnects N; therefore U contains no minimal disconnecting subset.

The following theorem gives a partial answer to the question: When does a disconnecting subset contain a minimal disconnecting subset?

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Proposition 1. *If X is a locally connected space and A is a disconnecting subset of X such that $X - A$ has some component U whose frontier does not properly contain the frontier of any component of $X - A$, A is closed, and A° (the interior of A) is empty, then A contains a minimal disconnecting subset.*

Proof. Since A is closed and X is locally connected, each component of $X - A$ is open; moreover, since $A^\circ = \emptyset$,

$$\bigcup \{\mathrm{Cl}\, W \mid W \text{ is a component of } X - A\} = X .$$

Let

$$K = \{x \in \mathrm{Fr}\, U \mid \text{some neighborhood of } x \text{ meets only } \mathrm{Cl}\, U\} .$$

Then

$$X - (\text{Fr } U - K) = (U \cup K) \cup \text{Ext } U,$$

$U \cup K$ is open and connected, and Ext U is open; therefore Fr $U - K$ disconnects X. We now show that Fr $U - K$ is a minimal disconnecting subset.

Case 1. Ext U is connected. Suppose $w \in \text{Fr } U - K$. If $U \cup K \cup \{w\}$ is relatively open in $(X - (\text{Fr } U - K)) \cup \{w\}$, then some neighborhood of w fails to meet Ext U, which, in turn, implies $w \in K$, a contradiction. On the other hand, if Ext $U \cup \{w\}$ is relatively open in $(X - (\text{Fr } U - K)) \cup \{w\}$, the some neighborhood of w fails to meet U; hence $w \notin \text{Fr } U$, again a contradiction. Therefore Ext $U \cup (U \cup K \cup \{w\})$ is connected: hence Fr $U - K$ is a minimal disconnecting subset.

Case 2. Ext U is not connected. Let V be a component of Ext U; then V is open. Therefore Fr $V \subseteq \text{Fr } U - K \subseteq \text{Fr } U$. This implies $K = \emptyset$ since Fr V is not a proper subset of Fr U. Therefore Fr U = Fr V for each component V of Ext U. This, in turn, implies that Fr U is a minimal disconnecting subset. Since U is open, Fr $U \subseteq A$.

The following example shows that a space can be homeomorphic to a compact, locally connected, connected subspace of R^3, yet still fail to have the property that every disconnecting subset contains a minimal disconnecting subset.

Example 2. Let

$$X_n = [0,1/n] \times [0,1], \quad n = 1, 2, 3, \ldots,$$

where each interval has its usual topology and each X_n has the product topology. Let Y be the identification space formed by identifying the points of the form $(x, 0)$ with the same first coordinate in each X_n and by identifying each point of the form $(0, y)$ with $(0, 0)$. Then Y is compact, connected and locally connected, and is easily embedded in R^3. However, the subset $\{(x, 0) \mid 0 \leq x \leq 1/2\}$ disconnects Y, but fails to contain a minimal disconnecting subset.

By "glueing" $[0, 1]$ onto Y by identifying 0 with $(0, 0)$ in Y, we obtain a space in which the subset $\{(x, 0) \mid 0 \leq x \leq 1/2\}$ fails to satisfy the hypotheses of Proposition 1, but which still contains the minimal disconnecting subset $\{(0, 0)\}$.

II. Some properties of disconnecting subsets

In this section we investigate the following properties:

A. Every disconnecting subset contains a closed disconnecting subset.

B. Every disconnecting subset contains a minimal disconnecting subset.

C. Every minimal disconnecting subset is closed.

Proposition 2. *For an arbitrary topological space X, the only implication which holds between the properties* A, B, *and* C *is* A *implies* C.

Proof. Example 2 shows that A does not imply B. The space $W = \{1, 2, 3\}$ with, topology $\{\emptyset, W, \{1, 2\}, \{2\}, \{2, 3\}\}$ demonstrates that B implies neither A nor C. The space of Example 1 shows that C fails to imply A. Proposition 3 below tells use that the space of Example 2 satisfies A, but not B. If a space X satisfies A and K is a minimal disconnecting subset of X, then K contains a closed disconnecting subset. But by the minimality of K, this subset must be K itself; hence K is closed. Therefore A implies C.

The following proposition proved in [2] implies that every completely normal space has property A.

Proposition 3. *Let X be a completely normal space. If a set E separates every pair of sets belonging to the system $A_1, \ldots, A_n$, then E contains a closed set F with the same property.*

III. Irreducible disconnections

Definition 1. *If K is a subset of a space X and $\mathscr{A}$ is a non-trivial partition of $X - K$ into subsets which are relatively open in $X - K$, we call the ordered pair $(K, \mathscr{A})$ a disconnection of K.*

Given two disconnections $(K, \mathscr{A})$ and $(K', \mathscr{A}')$, we say that $(K, \mathscr{A}) \leq (K', \mathscr{A}')$ if $K' \subseteq K$ and $\{U \cap (X - K) \mid U \in \mathscr{A}'\} = \mathscr{A}$.

It is easily confirmed that $\leq$ is a partial ordering of the set Δ of all disconnections

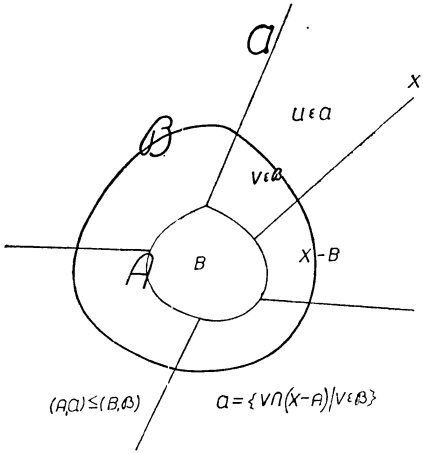

Fig. 1

of X. A maximal element of $\Delta, \leq$ is said to be an *irreducible disconnection.*

While a disconnecting subset need not contain a minimal disconnecting subset, the following theorem shows that in many instances a disconnection is related to an irreducible disconnection.

Proposition 4. *If A is a closed disconnecting subset of X, $A^\circ = \emptyset$, and $(A, \mathscr{A})$ is a disconnection of X, then there is an irreducible disconnection $(B, \mathscr{B})$ such that $(A, \mathscr{A}) \leq (B, \mathscr{B})$.*

Proof. For each $U \in \mathscr{A}$, set

$$U' = \mathrm{Cl}\, U - \mathrm{Cl}\,(\cup \{V \mid V \neq U, \quad V \in \mathscr{A}\}),$$

$$B = X - \cup \{U' \mid U \in \mathscr{A}\},$$

and

$$\mathscr{B} = \{U'\}, \quad U \in \mathscr{A}.$$

Claim 1. $(B, \mathscr{B})$ is a disconnection of X. Each U' is open. For it follows from the assumption that $A^\circ = \emptyset$ that

$$\mathrm{Cl}\, U \cup \mathrm{Cl}\,(\cup \{V \mid V \neq U, \quad V \in \mathscr{A}\}) = X,$$

and, hence $U' = X - \mathrm{Cl}(\cup \{V \mid V \neq U, V \in \mathscr{A}\})$. Also B is closed. Now $\cup \{\mathrm{Cl}\, V \mid V \neq U, V \in \mathscr{A}\} \subseteq \mathrm{Cl}(\cup \{V \mid V \neq U, \ V \in \mathscr{A}\})$ so that $U' \subseteq \mathrm{Cl}\, U - \cup \{\mathrm{Cl}\ V \mid V \neq U, \ V \in \mathscr{A}\}$, from which it follows that $U' \cap V' = \emptyset$ for any U and V in $\mathscr{A}$; therefore $(B, \mathscr{B})$ is a disconnection of X. Since $U \subseteq U'$ for all $U \in \mathscr{A}$ and $B \subseteq A$, we also have $(A, \mathscr{A}) < (B, \mathscr{B})$.

Claim 2. $(B, \mathscr{B})$ is irreducible. First, if $x \in B$, then x is in the closure of at least two members of $\mathscr{A}$. For if $x \in B \cap \mathrm{Cl}\, U$ for just $U \in \mathscr{A}$, then x has a neighborhood which meets U and this neighborhood then fails to meet $\cup \{V \mid V \neq U,\ V \in \mathscr{A}\}$, which implies $x \in U'$. On the other hand, $A^\circ = \emptyset$ implies that each neighborhood of x meets some $U \in \mathscr{A}$.

If $(B, \mathscr{B})$ is not irreducible, then there is a disconnection $(B', \mathscr{B}')$ with $(B, \mathscr{B}) < (B', \mathscr{B}')$. We may consider $\mathscr{B}' = \{U''\}$, $U \in \mathscr{A}$, with $U \subseteq U' \subseteq U''$. Now U'' is open in $X - B'$; hence there is an open set H such that $(X - B') \cap H \quad U''$. If $x \in U'' - U'$, then H meets at least two members of $\mathscr{A}$, which, in turn, leads to a contradiction of $(A, \mathscr{A}) \leq (B', \mathscr{B}')$.

Corollary. *If $(K, \mathscr{K})$ is a disconnection of X and K is closed, then there is an irreducible disconnection $(B, \mathscr{B})$ such that $(K, \mathscr{K}) \leq (B, \mathscr{B})$.*

Proof. Let $A = \mathrm{Fr}\, K$ and $\mathscr{A} = \mathscr{K} \cup \{K^\circ\}$. Then $(K, \mathscr{K}) \leq (A, \mathscr{A})$. By Proposition 4, there is an irreducible disconnection $(B, \mathscr{B})$ with $(A, \mathscr{A}) \leq (B, \mathscr{B})$; hence $(K, \mathscr{K}) \leq (B, \mathscr{B})$.

The techniques of Proposition 3 can be used in conjunction with Proposition 4 to prove:

Proposition 5. *If X is completely normal and $(A, \mathscr{A})$ is a disconnection of X such that $\mathscr{A}$ is finite, then there is an irreducible disconnection $(B, \mathscr{B})$ for which $(A, \mathscr{A}) < (B, \mathscr{B})$, and for which B is closed.*

The next proposition gives another characterization of the set U' in the proof of Proposition 4.

Proposition 6. *Let the hypotheses and notation be as in Proposition 4 and its proof. Then for each $U \in \mathscr{A}$,*

$$U' = \cup \{W \mid W \in \mathscr{D},\quad U \subseteq W,\quad (A, \mathscr{A}) \leq (D, \mathscr{D})\}\,.$$

Proof. Let $H = \cup \{W \mid W \in \mathscr{D},\ U \subseteq W,\ (A, \mathscr{A}) \leq (D, \mathscr{D})\}$. Since $(A, \mathscr{A}) \leq (D, \mathscr{D})$ and $U \subseteq U'$, $U' \subseteq H$. We now show $H \subseteq U'$. Suppose $x \in H$. Then $x \in W$ where $U \subseteq W$, $(A, \mathscr{A}) \leq (D, \mathscr{D})$, and $W \in \mathscr{D}$. We will show $x \in \mathrm{Cl}\, U - \mathrm{Cl}(\cup \{V \mid U \neq V,\ V \in \mathscr{A}\})$. If $x \notin U$, then $x \in A$. In that case, there is an open set G which contains x and meets only U since $x \in W = (X - D) \cap G$ for an appropriate open set G. Consequently, $x \notin \mathrm{Cl}(\cup \{V \mid V \neq U,\ V \in \mathscr{A}\})$. But then each neighborhood of x must meet U or A would not be empty. Therefore $x \in \mathrm{Cl}\, U - \mathrm{Cl}(\cup \{V \mid U \neq V,\ V \in \mathscr{A}\}) - U'$.

In the previous section we proved that a minimal disconnecting subset need not be closed. We now discuss conditions under which a disconnecting subset associated with an irreducible disconnection is closed.

Proposition 7. If $(A, \mathscr{A})$ is an irreducible disconnection of X, then A is closed if and only if $A^\circ = \emptyset$.

Proof. Suppose $A^\circ \neq \emptyset$, but A is closed. Set $B = A - A^\circ$ and $\mathscr{B} = \mathscr{A} \cup \{A\ \}$. Then $(B, \mathscr{B})$ is a disconnection with $(A, \mathscr{A}) < (B, \mathscr{B})$, contradicting the maximality of $(A, \mathscr{A})$.

Suppose now that A is not closed. Then there are $U_1, U_2 \in \mathscr{A}$ such that $U_1 - V_1 \cap \cap (X - A)$ and $U_2 = V_2 \cap (X - A)$, where V_1 and V_2 are open sets and $\emptyset \neq V_1 \cap V_2 \subseteq \subseteq A^\circ$. For if such V_1 and V_2 fail to exist, then for each $U_i \in \mathscr{A}$, we can find V_i, an open set. such that $U_i = V_i \cap (X - A)$ for which $V_i \cap V_{i'} = \emptyset$, $i \neq i'$. Take $B = X - \cup_{\mathscr{A}} V_i$

and $\mathscr{B}$ $\{V_i\}$. Then $(A, \mathscr{A}) < (B, \mathscr{B})$, contradicting the maximality of $(A, \mathscr{A})$. Therefore $A^\circ \neq \emptyset$.

Corollary. *If X is T_3 and $(A, \mathscr{A})$ is an irreducible disconnection of X, then A is closed.*

Proof. If A is not closed, then $A^\circ \neq \emptyset$. Take $x \in A^\circ$. There is a neighborhood V of x such that $x \in V \subseteq \mathrm{Cl}\, V \subseteq A^\circ$. Set $B = A - \mathrm{Cl}\, V$ and $\mathscr{B} = \mathscr{A} \cup \{A^\circ - \mathrm{Cl}\, V\}$. Then $(A, \mathscr{A}) < (B, \mathscr{B})$, contradicting the maximality of $(A, \mathscr{A})$.

If $(A, \mathscr{A})$ is a disconnection and A is a minimal disconnecting subset, then $(A, \mathscr{A})$ is necessarily irreducible; consequently, any minimal disconnecting subset is always associated with some irreducile disconnection. As a second corollary of Proposition 7 we may therefore conclude:

Corollary. *If X is T_3 and A is a minimal disconnecting subset of X, then A is closed.*

IV. Decomposition theories

The general structure of the partially ordered family of decompositions of a topological space may be abstracted to give a structure which occurs in more than just topological contexts.

Definition 2. *A collection Δ of ordered pairs $(A, \mathscr{A})$ is said to be a decomposition theory for a set X if:* 1) $(A, \mathscr{A}) \in \Delta$ implies $A \subseteq X$ *and $\mathscr{A}$ is a partition of $X - A$.* 2) $(A, \{X - A\}) \in \Delta$ *for each* $A \subseteq X$. 3) *If $(A, \mathscr{A})$ and $(B, \mathscr{B})$ are in Δ, then* $(A \cup B, \{U \cap V \mid U \in \mathscr{A}, V \in \mathscr{B}\}) \in \Delta$. 4) *If $(A, \mathscr{A}) \in \Delta$ and $\mathscr{A}'$, is a partition of $X - A$ such that $\mathscr{A}$ refines $\mathscr{A}'$, then $(A, \mathscr{A}') \in \Delta$.*

Let Δ be a decomposition theory on the set X. We say that $(A, \mathscr{A})$ is a *disconnection* of X if $\mathscr{A}$ contains at least two non-empty subsets of X. If $(A, \mathscr{A})$ and $(B, \mathscr{B})$ are any two elements of Δ, we let $(A, \mathscr{A}) \leq (B, \mathscr{B})$ if $B \subseteq A$ and $\{U \cap (X - A) \mid U \in \mathscr{B}\} = \mathscr{A}$. Then $<$ is a partial ordering of Δ. A disconnection which is maximal in $\Delta, \leq$, is said to be *irreducible.*

Example 3. Let X, τ be a topological space. Then $\Delta_\tau = \{(A, \mathscr{A}) \mid A \subseteq X,\ \mathscr{A}$ is a partition of $X - A$ into relatively open subsets$\}$
is a decomposition theory for X. Also

$$\tilde{\Delta}_\tau = \{(A, \mathscr{A}) \in \Delta_\tau \mid \mathscr{A} \text{ is finite}\}$$

is a decomposition theory for X.

The following is easily proved.

Proposition 8. *If $\{\Delta_i\}$, $i \in I$, is a non-empty family of decomposition theories on a set X, then $\cap_I \Delta_i$ is also a decomposition theory for X.*

Corollary. *Suppose $\mathscr{J}$ is a family of ordered pairs $(A, \mathscr{A})$ such that $A \subseteq X$ and $\mathscr{A}$ is a partition of $X - A$. Then there is a unique minimal decomposition theory $\Delta \mathscr{J}$ such that $\mathscr{J} \subseteq \Delta_{\mathscr{J}}$*

Proof. The family $\mathscr{J}$ is at least a subset of the *discrete* decomposition theory $\{(A, \mathscr{A}) \mid A \subseteq X, \mathscr{A}$ is a partition of $X - A\}$. Then $\Delta_{\mathscr{J}}$ is the intersection of all decomposition theories for X which contain $\mathscr{J}$.

We now give further examples of decomposition theories.

Example 4. Let $G, \neq$ be a group and $\mathscr{S}$ be any family of subgroups of G. Then there is a unique minimal decomposition theory which contains $\{(\emptyset, G/H \mid H \in \mathscr{S}\}$.

Example 5. Let π be the Euclidean plane and π_L and π_L be the two sides of any line L of π. Then $\{(L, \{\pi_L, \pi_L\}) \mid L$ is a line of $\pi\}$ generates a decomposition theory for π.

A decomposition theory generates a topology in the following manner.

Definition 3. *Let Δ be a decomposition theory for X. For each $(A, \mathscr{A}) \in \Delta$ and $V \in \mathscr{A}$, set*

$$U(V) = \cup \{W \mid W \in \mathscr{B},\ (B, \mathscr{B}) \in \Delta,\ V \subseteq W,\ (A, \mathscr{A}) < (B, \mathscr{B})\}.$$

The family of all such $U(V)$ obtained from all members of Δ serves as the subbasis for a topology on X; we denote this topology by τ_Δ.

We may think of τ_Δ as the coarsest topology τ on X for which $\Delta \subseteq \Delta_\tau$.

While a topology generates two decomposition theories, and a decomposition theory generates a topology, the concepts do not appear to be interchangeable. The following example shows that for a topological space X, τ, τ and τ_{Δ_τ} need not be related by inclusion.

Example 6. Let $X = \{1, 2, 3\}$ with $\tau = \{X, \emptyset, \{1, 2\}, \{2, 3\}, \{2\}\}$. In this instance, $\tau_{\Delta\tau} = \{X, \emptyset, \{1\}, \{3\}, \{1, 3\}\}$.

The next propositions are concerned with relationships between τ and τ_{Δ_τ}.

Proposition 9. *Let X, τ be a topological space and $\mathscr{S}$ be the subbasis for τ_{Δ_τ} described in Definition 3. Then if U is a member of such that* Fr U *disconnects X, $U \in \mathscr{S}$.*

Proof. If Fr U disconnects X, then (Fr U, $\{U$, Ext $U\}) \in \Delta_\tau$, Fr U is closed, and (Fr $U)^\circ = \emptyset$. It therefore follows from Proposition 6 that $U' = $ Cl U − Cl(Ext U) $= U \in \mathscr{S}$.

Corollary. *If X has a basis of neighborhoods whose frontiers disconnect X, then $\tau \subseteq \tau_{\Delta_\tau}$.*

Lemma 2. *The space X, τ has a basis of open sets whose frontiers disconnect X if and only if each point of X has a neighborhood whose frontier disconnects X.*

Proof. Clearly if X has such a basis, then each point has a neighborhood whose frontier disconnects X. Assume now that each point x has some neighborhood V whose frontier disconnects X. Then if U is a neighborhood of x, then $U \cap V$ is a neighborhood of x whose frontier disconnects X and which is a subset of U. Consequently, there is an open neighborhood system, and hence a basis for τ, which consists entirely of open sets whose frontiers disconnect X.

In any T_2-space X of more than one point, every point has a neighborhood whose frontier disconnects X. We can therefore state:

Proposition 10. *If X, τ is T_2, or is not connected, then $\tau \subseteq \tau_{\Delta\tau}$.*

The following example shows that τ may be a proper subset of $\tau_{\Delta\tau}$.

Example 7. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$, and τ have as basis $\mathscr{B} = \{\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 7\}, \{1\}, \{3\}, \{5\}, \{7\}\}$. Each point of X is contained in a member of $\mathscr{B}$ whose frontier disconnects X; hence $\tau \subseteq \tau_{\Delta\tau}$. But from the member $(\{3, 5\}, \{\{1, 2\}, \{4\}, \{6, 7\}\})$ of Δ_τ, we find $\{1, 2\} \in \tau_{\Delta\tau}$, but not in τ.

Proposition 11. If X, τ has the property that for every disconnection $(A, \mathscr{A})$ there exists $(B, \mathscr{B}) \in \Delta_\tau$ with B closed such that $(A, \mathscr{A}) < (B, \mathscr{B})$, then $\tau_{\Delta\tau} \subseteq \tau$.

Proof. We may restrict our attention to those members $(A, \mathscr{A})$ of Δ for which A is closed, and thus for which the elements of $\mathscr{A}$ are open sets. It follows then that the subbasis elements for $\tau_{\Delta\tau}$ are all members of τ; hence $\tau_{\Delta\tau} \subseteq \tau$.

Using Proposition 5 together with Proposition 11, we can prove:

Proposition 12. *If X, τ is completely normal, then $\tau_{\tilde{\Delta}_\tau} \subseteq \tau$. Since $\tau_{\tilde{\Delta}_\tau} \subseteq \tau_\Delta$,* Propositions 11 and 12 give:

Proposition 13. *If X, τ is completely normal, then $\tau_{\tilde{\Delta}_\tau} = \tau$.*

Corollary. *If X, τ is metrizable, then $\tau_{\tilde{\Delta}_\tau} = \tau$.*

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