Matematický časopis

Anna Neubrunnová

On certain generalizations of the notion of continuity

Matematický časopis, Vol. 23 (1973), No. 4, 374--380

Persistent URL: http://dml.cz/dmlcz/126571

Matematický časopis 23 (1973), No. 4

ON CERTAIN GENERALIZATIONS OF THE NOTION OF CONTINUITY

ANNA NEUBRUNNOVÁ, Bratislava

We shall deal with mutual connections among various generalizations of the notion of a continuous function. The generalizations were introduced by different authors and it turns out that sometimes different definitions give the same result, at least for certain types of spaces.

The notions which will be dealt with are quasi-continuity, cliquishness, semi-continuity, simple continuity, pseudo-continuity and almost continuity.

1. Semi-continuity and quasi-continuity

In what follows X and Y denote topological spaces. The notion of the quasi-continuous function was introduced by S. Kempisty [1]. An equivalent definition was given by N. W. Bledsoe [2], where the quasi-continuous function is called neighbourly. The equivalency was proved by S. Marcus [3]. One of the definitions is as follows.

The function $f : X \to Y$ is said to be quasi-continuous at the point $x \in X$ if for any open neighbourhood V of x and any open neighbourhood G of $f(x)$ there exists a non-empty open set $U \subset V$ such that $f(U) \subset G$.

We shall prove that the notion of the quasi-continuous function is equivalent with that of a semi-continuous function as defined by N. Levine [4] by means of the notion of the semi-open set. A set $A \subset X$ is said to be semi-open provided that there is an open set O such that $O \subset A \subset \bar{O}$ ($\bar{O}$ denotes the closure of O).

The function $f : X \to Y$ is called semi-continuous if for any open set $G \subset Y$ the set $f^{-1}(G)$ is a semi-open set.

Theorem 1.1. *Let $f : X \to Y$. Thus f is semi-continuous if and only if it is quasi-continuous (i. e. quasi-continuous at every point $x \in X$).*

Proof. Let f be semi-continuous. Let $x_0 \in X$ and G any open set such that $f(x_0) \in G$. Let V be any open set containing x_0. Under the assumption $f^{-1}(G)$ is a semiopen set. Hence

$$\overline{\operatorname{int} f^{-1}(G)} \supset f^{-1}(G)$$

according to [4, Theorem 1]. Put $U = V \cap \operatorname{int} f^{-1}(G)$. Since $x_0 \in f^{-1}(G) \subset \subset \overline{\operatorname{int} f^{-1}(G)}$, there is a point in V belonging to $\operatorname{int} f^{-1}(G)$, hence U is a non-empty set. Evidently $U \subset V$ and $f(U) = f(V \cap \operatorname{int} f^{-1}(G)) \subset f(f^{-1}(G)) \subset G$.

Hence f is quasi-continuous at x_0. Since x_0 was arbitrarily chosen, the function f is quasi-continuous on X.

Conversely, let f be quasi-continuous on X. Choose any non-empty open set G. Let $x_0 \in f^{-1}(G)$ and let V be any open set containing x_0. The quasi-continuity of f at x_0 implies the existence of $U \neq \emptyset$, $U \subset V$, U open such that $f(U) \subset G$, $U \subset f^{-1}(G)$. Thus $U \subset \operatorname{int} f^{-1}(G)$. Hence $\emptyset \neq U = V \cap U \subset \subset V \cap \operatorname{int} f^{-1}(G)$. It means that for any open set V containing x_0

$$V \cap \operatorname{int} f^{-1}(G) \neq \emptyset \, .$$

Hence $x_0 \in \overline{\operatorname{int} f^{-1}(G)}$. Since the point x_0 was arbitrarily chosen in $f^{-1}(G)$ we have $f^{-1}(G) \subset \overline{\operatorname{int} f^{-1}(G)}$. Thus $f^{-1}(G)$ is semi-open.

The notion of semi-continuity may be formulated also for a point. Thus a function $f : X \to Y$ will be called semicontinuous at the point $x \in X$ if for any open set $G \subset Y$ such that $f(x) \in G$ there is a semi-open set U such that $x \in U$ and $f(U) \subset G$.

A function f is semi-continuous on X if and only if it is semi-continuous at every point $x \in X$. The proof of this assertion gives Theorem 12 in [4]. (Let us remark that the notion of semi-continuity at the point is not explicitely introduced in [4]).

A question arises whether the semi-continuity at a point is equivalent to the quasi-continuity of the function at that point.

The fact that semi-continuity at a point x implies quasi-continuity at x can be proved in the same way as the first part of Theorem 1.1. The second part of Theorem 1.1 uses quasi-continuity at every point $x \in X$. But the proof of the last may be modified so that the following theorem holds.

Theorem 1.2. *The function $f : X \to Y$ is quasi-continuous at $x_0 \in X$ if and only if it is semi-continuous at x_0.*

Proof. From what was said it follows that it is sufficient to prove that quasi-continuity at x_0 implies semicontinuity at x_0.

Thus let G be open so that $f(x_0) \in G$. Let V be open containing x_0. There exists an open $U \subset V$ such that $\emptyset \neq U$ and $f(U) \subset G$. The union of all those U for all open V containing x_0 is an open set W. Put $S = W \cup \{x_0\}$. Evidently $f(S) \subset G$. But the set S is semiopen. The proof is finished.

It is well known that the notion of the semi-continuity is different from that of the upper and lower semi-continuity according to which the real function f is said to be lower semi-continuous at x_0 provided that for any $a < f(x_0)$ there exists a U open such that $x_0 \in U$ and for any $x \in U$ $f(x) > a$.

The upper semi-continuity is defined analogously.

Example 1.1. (See also [4]). Let $X = Y = \langle 0; 1\rangle$. Define $f: X \to Y$ as follows:

$$f(x) = \begin{cases} 0 \text{ if } x \in \langle 0; 1/2) \cup (1/2; 1\rangle \\ 1 \text{ if } x = 1/2 \end{cases}$$

f is not semi-continuous but it is upper semi-continuous at any $x \in \langle 0; 1\rangle$.

Example 1.2. Put $X = \langle 0, 1\rangle$, $Y = (-\infty, \infty)$. Consider the following sequence of intervals: $I_1 = \left\langle \frac{1}{2}, 1 \right\rangle, I_2 = \left(\frac{1}{3}, \frac{1}{2}\right), I_3 = \left\langle \frac{1}{4}, \frac{1}{3}\right\rangle, I_4 = \left(\frac{1}{5}, \frac{1}{4}\right)$, $I_5 = \left\langle \frac{1}{6}, \frac{1}{5}\right\rangle, I_6 = \left(\frac{1}{7}, \frac{1}{6}\right)$ etc. Evidently $\bigcup_{n=1}^{\infty} I_n = (0, 1\rangle$. Define the function f on $\langle 0, 1\rangle$ in the following way: $f(x) = 1$ if $x \in \left\langle \frac{1}{2}, 1\right\rangle, f(x) = -1$ if $x \in \left(\frac{1}{3}, \frac{1}{2}\right)$, $f(x) = \frac{1}{2}$ if $x \in \left\langle \frac{1}{4}, \frac{1}{3}\right\rangle$, $f(x) = 1$ if $x \in \left(\frac{1}{5}, \frac{1}{4}\right)$, $f(x) = -1$ if $x \in \left\langle \frac{1}{6}, \frac{1}{5}\right\rangle$, $f(x) = \frac{1}{2}$ if $x \in \left(\frac{1}{7}, \frac{1}{6}\right)$, etc., so that the function will take the value 1 on $I_1, I_4, I_7, I_{10}, \ldots$, the value -1 on $I_2, I_5, I_8, I_{11}, \ldots$ and the value $\frac{1}{2}$ on $I_3, I_6, I_9, I_{12}, \ldots$ Further put $f(0) = 1/2$. The function f is neither upper, nor lower semi-continuous at the point 0. But it is evidently quasi-continuous, hence semi-continuous at any $x \in \left\langle 0, \frac{1}{2}\right\rangle$.

2. Simple continuity and cliquishness

The notion of the cliquish function was introduced as a neighbourly function by W. W. Bledsoe [2] for the functions of real variable taking the values in a metric space. It was generalized by Thielman [5] for the functions defined on a topological space taking values in a metric space.

The function $f: X \to Y$, where X is a topological space and Y a metric space with the metric ϱ, is said to be cliquish at the point $x \in X$ provided that for any $\varepsilon > 0$ and any neighbourhood U of x there exists a non-empty open set $G \subset U$ such that for any $x_1, x_2 \in G$ $\varrho f(x_1), f(x_2)] < \varepsilon$.

Evidently the above notion is more general than that of a continuous and quasi-continuous function. Simple examples showing that a cliquish function need not be quasi-continuous are well-known.

In paper [6] the notion of the simply-continuous function was introduced by means of the notion of the simply open set.

A set A is said to be simply open if $A = O \cup N$, where O is open and N a nowhere dense set. A function $f: X \to Y$ (X, Y topological spaces) is called simply continuous if for any open $G \subset Y$ $f^{-1}(G)$ is a simply open set.

Since any semi-open set is simply open [6, Theorem 1.1.1] it follows that any semi-continuous function is simply continuous. The converse is not true. [6, Example 1.1.2].

Theorem 2.1. *Let X be a topological space of the second category at each of its points and Y a separable metric space. Then a function $f: X \to Y$ which is simply-continuous is also cliquish.*

Proof. Let $\varepsilon > 0$, $x_0 \in X$ and V any open set containing x_0. Let $\{y_n\}$ be a countable dense set in Y. Denote by $K(y_n, \varepsilon/2)$ the open sphere with the centre y_n and the radius $\varepsilon/2$. Since $\bigcup_{n=1}^{\infty} K(y_n, \varepsilon/2) = Y$ we have $[\bigcup_{n=1}^{\infty} f^{-1}(K(y_n, \varepsilon/2))] \cap V = V$. Under the assumption $f^{-1} K(y_n, \varepsilon/2)] = G_n \cup Z_n$, where G_n is open and Z_n nowhere dense in X. Thus

$$[(\bigcup_{n=1}^{\infty} G_n) \cap V] \cup [\bigcup_{n=1}^{\infty} Z_n \cap V] = V .$$

Since $\bigcup_{n=1}^{\infty} Z_n \cap V$ is of the first category in V and V is of the second category, we have $\bigcup_{n=1}^{\infty} G_n \cap V \neq \emptyset$. Hence there is n_0 such that $G_{n_0} \cap V \neq \emptyset$. Putting $U = G_{n_0} \cap V$, we have an open set $U \subset V$ such that for any $x_1, x_2 \in U$, $f(x_1) \in K(y_{n_0}, \varepsilon/2)$, $f(x_2) \in K(y_{n_0}, \varepsilon/2)$, so $\varrho(f(x_1), f(x_2)) < \varepsilon$. Thus f is cliquish at the point x_0.

The converse of the above theorem is not true.

Example. Let $X = Y = \langle 0, 1\rangle$. Let f be the Riemann function

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \in \langle 0, 1\rangle,\ p, q \text{ are relatively prime integers, } q > 0 \\ 0 & \text{if } x \in \langle 0, 1\rangle \text{ is irrational.} \end{cases}$$

Thus f is cliquish. It is not simply continuous because if $G = (0, 1\rangle$, then $f^{-1}(G)$ is the set of all rational numbers which is not simply open.

The proof of Theorem 2.1 uses to a great extent the fact that X is of the second category at each of its points. Nevertheless it may be shown that this assumption is not essential at least for those cases where the metric space Y has suitable properties.

Theorem 2.2. *Let X be a topological space and Y a totally bounded metric space. Then any function $f\colon X \to Y$ which is simply continuous on X is cliquish on X.*

Proof. Let $\varepsilon > 0$. Let $x \in X$ and U any neighbourhood of x. Since Y is totally bounded there exists a finite number $G_1, G_2, \ldots, G_n$ of open sets with the diameters less then ε and such that $Y = \bigcup_{i=1}^{n} G_i$. We have $f^{-1}(G_i) = U_i \cup Z_i$, where U_i are open and Z_i nowhere dense in X. It is sufficient to prove that there exists i_0 such that $U_{i_0} \cap U \neq \emptyset$. But the latter is true because $U = X \cap U = f^{-1}(Y) \cap U = [\bigcup_{i=1}^{n} f^{-1}(G_i)] \cap U = [\bigcup_{i=1}^{n} U_i \cap U] \cup [\bigcup_{i=1}^{n} Z_i \cap U]$. The set $\bigcup_{i=1}^{n} Z_i \cap U$ is nowhere dense in U. Hence $\bigcup_{i=1}^{n} U_i \cap U \neq \emptyset$. Thus $U_{i_0} \cap U \neq \emptyset$ for some i_0.

3. Pseudo-continuity and some other types of continuities

The notion of the simply continuous function may be generalized by means of the pseudo-open set to the notion of the pseudo-continuous function.

A set A in a topological space X is said to be pseudo-open if $A = O \cup N$, where O is open and N is of the first category. A function $f : X \to Y$ (X, Y topological spaces) is said to be pseudo-continuous on X if for any open set $G \subset Y$ the set $f^{-1}(G)$ is pseudo-open.

Since any simply open set is pseudo-open, it follows that any simply continuous function is pseudo-continuous. The converse is not true. For the proof it is sufficient to take the Riemann function.

The following theorem is a generalization of Theorem 2.1 and may be proved by the same method as the latter.

Theorem 3.1. *Let X be a topological space of the second category at each of its points and Y a separable metric space. Let $f : X \to Y$ be pseudo-continuous. Then f is cliquish.*

The converse of the last theorem may be proved in a more general form.

Theorem 3.2. *Let X be topological and Y a metric space. Let $f : X \to Y$ be cliquish. Then f is pseudo-continuous.*

Proof. If f is cliquish, then the set of its points of discontinuity is of the first category in X [**7**]. Hence f is pseudo-continuous [6, Theorem 1.1.10].

Note that Theorem 1.1.10 in paper [6] is formulated for the case where Y is a separable metric space and gives a necessary and sufficient condition for f to be pseudocontinuous. Nevertheless the part of the theorem which was used in our proof does not use the separability of Y.

Theorem 3.3. *Let X be of the second category in each of its points and Y a separable metric space. Then $f: X \to Y$ is pseudo-continuous if and only if f is cliquish.*

Proof. A consequence of Theorems 3.1 and 3.2.

Of course there are many other types of continuities and the types which we have discussed here do not exhaust all which are used. We shall mention another type which is sometimes studied.

In [8] and [9] almost continuous functions were studied.

The function $f: X \to Y$ (X, Y topological spaces) is said to be almost continuous at the point $x \in X$ provided that for any open set $G \subset Y$ with $f(x) \in G$ we have $x \in \overline{\operatorname{int} f^{-1}(G)}$.

The almost continuous function need not be pseudocontinuous and hence need not belong to any of the foregoing types.

Example 3.1. Let $X = Y = \langle 0, 1 \rangle$.

$$\text{Let } f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Evidently f is almost continuous. It is not pseudocontinuous because if G $\langle 0, 1/2)$, then $f^{-1}(G)$ is the set of all irrational numbers in $\langle 0, 1 \rangle$ which is not pseudo-open.

To prove that none of the foregoing types of continuity imply the almost continuity it is sufficient to show that the quasi-continuity does not imply the almost continuity.

Example 3.2. Let $X = Y = \langle 0, 1 \rangle$.

$$\text{Put } f(x) = \begin{cases} 0 & \text{if } x \in \langle 0, 1/2 \rangle \\ 1 & \text{if } x\ (1/2, 1 \rangle. \end{cases}$$

f is quasi-continuous at any point $x \in \langle 0, 1 \rangle$ but it is not almost continuous at the point $x = 1/2$.

REFERENCES

[1] KEMPISTY, S.: Sur les fonctions quasicontinues. Fundam. math., *19*, 1932, 184 197.

[2] BLEDSOE, W. W.: Neighbourly functions. Proc. Amer. Math. Soc. *3*, 1952, 114—115.

[3] MARCUS, S.: Sur les fonctions quasicontinues au sens de S. Kempisty. Collog. math., 1961, 47 53.

[4] LEVINE, N.: Semi-open sets and semi-continuity in topological spaces. Amer. Math. Monthly, *70*, 1963, 36—43.

[5] THIELMAN, H. P.: Types of functions. Amer. Math. Monthly, *60*, 1953, 156 161.

[6] BISWAS, N.: On some mappings in topological spaces. Thesis submitted to the University of Calcutta for the D. Phil. Arts Degree. 1970.
[7] NEUBRUNNOVÁ, A.: On quasi-continuous and cliquish functions. (To be published)
[8] HUSAIN, T.: Almost continuous mappings. Ann. Soc. Math. Polonaise, Série 1, Prace matemat., *10*, 1966, 1–7.
[9] HUSAIN, T.: Some remarks about real almost continuous functions. Math. Mag., *40*, 1967, 250–254.

Received August 2, 1972

Katedra matematickej analýzy
Prírodovedeckej fakulty Univerzity Komenského
Bratislava