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Matematický časopis, Vol. 23 (1973), No. 4, 352--355

Persistent URL: http://dml.cz/dmlcz/126569

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NOTE ON MODULAR AND DISTRIBUTIVE EQUALITIES IN LATTICES

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In the paper [4] F. Šik studies a sublattice $\langle a, b, c\rangle$ of a lattice S that is generated by the triple of the elements $a, b, c \in S$. He investigates the properties of this sublattice when instead of some modular or distributive identity in the lattice S there holds a corresponding equality only for the triple a, b, c. F. Šik considers the following equalities:

Modular:

(1) $(a \vee b) \wedge c = a \vee (b \wedge c)$, where $a \leqq c$,

(1*) $(a \wedge b) \vee c = a \wedge (b \vee c)$, where $a \geqq c$,

(2) $a \vee [b \wedge (c \vee a)] = (a \vee b) \wedge (a \vee c)$,

(2*) $a \wedge [b \vee (c \wedge a)] = (a \wedge b) \vee (a \wedge c)$.

Distributive:

(3) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

(3*) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(4) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$.

In this paper we shall study some other equalities:

(5) $[a \vee (b \wedge c)] \wedge (b \vee c) = [a \wedge (b \vee c)] \vee (b \wedge c)$,

(6) $(b \vee c) \wedge [a \vee (b \wedge c)] = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$,

(6*) $(b \wedge c) \vee [a \wedge (b \vee c)] = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$,

(7) $(b \wedge c) \vee [a \wedge (b \vee c)] = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$,

(7*) $(b \vee c) \wedge [a \vee (b \wedge c)] = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$.

It is easy to prove that a lattice S is modular if and only if there holds the equality (5) for all triples of elements in S (otherwise if the identity (5) is satisfied in S).

Similarly a lattice S is distributive if and only if the identity (6) or (6*) is satisfied in S.

Now, let x, y, z be elements of a lattice S. The symbol xyz expresses that the following is satisfied:

$$(x \wedge y) \vee (y \wedge z) = y = (x \vee y) \wedge (y \vee z) .$$

Then for $a, b, c \in S$ we shall denote

$$B(a, b, c) = \{x \in S: axb, bxc, cxa\} .$$

There holds

Theorem 1. *For elements a, b, c of a lattice S the equality* (5) *is satisfied if and only if $B(a, X, Y) \neq \emptyset$, where X is a left-hand side, Y is a right-hand side of the equality* (5).

Proof. 1. If the equality (5) is satisfied, then clearly $X \in B(a, X, Y)$.

2. Assume that (5) is not satisfied. Therefore

$$[a \wedge (b \vee c)] \vee (b \wedge c) < [a \vee (b \wedge c)] \wedge (b \vee c) .$$

Let us denote

$$p = b \wedge c,\ q = a,\ r = b \vee c .$$

Then

$$p \leqq r, p \vee (q \wedge r) = (b \wedge c) \vee [a \wedge (b \vee c)] < [(b \wedge c) \vee a] \wedge (b \vee c) = \\ = (p \vee q) \wedge r .$$

Now, we denote

$$\begin{aligned} Y &= p \vee (q \wedge r) = (b \wedge c) \vee [a \wedge (b \vee c)] , \\ X &= (p \vee q) \wedge r = [(b \wedge c) \vee a] \wedge (b \vee c) , \\ D &= q \wedge r = a \wedge (b \vee c) , \\ E &= p \vee q = a \vee (b \wedge c) . \end{aligned}$$

From [1, II, 9, proof of Theorem 9.3] it follows that $E > X > Y > D$. $E > q > D$, $X \parallel q \parallel Y$ form a "pentagonal" sublattice of S.

Now, let us suppose that there exists $z \in B(a, X, Y)$. Thus

$$(X \wedge z) \vee (z \wedge Y) = z = (X \vee z) \wedge (z \vee Y) .$$

From $Y < X$ it follows $X \wedge z = z$, $z = z \vee Y$, then it is $Y \leqq z \leqq X$. Thus

$$X = E \wedge X = (a \vee z) \wedge (z \vee X) = z = (a \wedge z) \vee (z \wedge Y) = D \vee Y = Y,$$

a contradiction.

Next we shall study the distributivity of the lattice $\langle a, b, c\rangle$.

Lemma. *Let the equalities* (6), (6*), *or equalities arising from* (6), (6*) *by some permutation of the elements a, b, c, be satisfied. Then also the corresponding equality* (5) *and the equality* (4) *are satisfied for a, b, c.*

Proof. In an arbitrary lattice S there are satisfied $Y \leqq X, t \leqq t^*$, where t is a left-hand side, t^* is a right-hand side of the equality (4). If now

$$t = X,\ t^* = Y ,$$

then

$$t^* = Y \leqq X = t ,$$

thus

$$X = Y,\ t = t^* .$$

Let us state the following conditions:

(I) There are satisfied all six distributive equalities that we can obtain from (3), (3*) by permutations of a, b, c.

(II) There is satisfied one of the following conditions:

(i) $\langle a, b, c\rangle$ satisfies the upper covering condition.

(ii) $\langle a, b, c\rangle$ satisfies the lower covering condition.

(iii) $\langle a, b, c\rangle$ is semimodular.

(III) There is satisfied one of six distributive equalities that we can obtain from (6), (6*) by permutations of a, b, c.

Now, the following theorem holds:

Theorem 2. *Let S be a lattice, $a, b, c \in S$. Then the following conditions are equivalent:*

(*a*) *There are satisfied* (4) *and (II).*

(*b*) *There are satisfied (I) and* $B(a, b, c) \neq \emptyset$.

(*c*) $\langle a, b, c\rangle$ *is distributive.*

(*d*) $\langle a, b, c\rangle$ *is modular and one of seven equalities* (3), (3*), (4) *is satisfied for* a, b, c.

(*e*) *There are satisfied (I) and (III).*

(*f*) $\langle a, b, c\rangle$ *is modular and (III) is satisfied.*

(*g*) *There are satisfied (II) and one of three couples of mutually dual equalities that can be obtained from* (6), (6*) *by permutations of* a, b, c.

(*h*) $\langle a, b, c\rangle$ *is modular and one of six equalities* (7), (7*) *is satisfied for* a, b, c.

Proof. Equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (*d*) are proved in [4].

(e) $\Rightarrow$ (c):

By (6)

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (b \vee c) \wedge [a \vee (b \wedge c)] .$$

By (I)

$$(b \vee c) \wedge [a \vee (b \wedge c)] = (b \vee c) \wedge (a \vee b) \wedge (a \vee c) .$$

Therefore (I) and (4) are satisfied and hence (c) holds by [2].

(c) $\Rightarrow$ (e): Evident.

(f) $\Rightarrow$ (d): Let us denote

$$t = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) ,$$
$$v = (a \vee b) \wedge [c \vee (a \wedge b)] .$$

Let $t = v$ hold. Then

$$c \wedge t = c \wedge \{(a \wedge b) \vee [(b \wedge c) \vee (c \wedge a)]\} .$$

Since $\langle a, b, c\rangle$ is modular,

$$c \wedge t = [c \wedge (a \wedge b)] \vee [(b \wedge c) \vee (c \wedge a)] = (c \wedge a) \vee (c \wedge b) .$$

Furthermore

$$c \wedge v = c \wedge \{(a \vee b) \wedge [c \vee (a \wedge b)]\} = c \wedge (a \vee b) .$$

Since $t = v$, $c \wedge t = c \wedge v$; and hence

$$c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b) .$$

(c) $\Rightarrow$ (f): Evident.
(g) $\Rightarrow$ (a): By Lemma $t = t^*$.
(c) $\Rightarrow$ (g): Evident.
(h) $\Rightarrow$ (f): (7) is satisfied, thus

$$t = (b \wedge c) \vee [a \wedge (b \vee c)] .$$

Now, by modular identity (5) there holds

$$t = (b \vee c) \wedge [a \vee (b \wedge c)] .$$

(c) $\Rightarrow$ (h): Evident.

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Received June 14, 1972

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