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Eliška Tomová

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ON THE DECOMPOSITION OF THE COMPLETE DIRECTED GRAPH INTO FACTORS WITH GIVEN DIAMETERS

ELIŠKA TOMOVÁ, Bratislava

The authors of paper [1] study the problem of the decompositions of the complete undirected graph $\langle n \rangle$ into m factors with given diameters. In the present paper we study a similar problem for the complete directed graph with respect to two factors.

All graphs in the present paper are directed, without loops and between two vertices of the graph there exist at most two edges with opposite directions. The complete directed graph G with n vertices will be denoted by $\langle\langle n \rangle\rangle$ and we mean by it the graph with n vertices, two arbitrary different vertices of which are connected with just two edges with opposite directions. By a factor of a directed graph G we mean an arbitrary subgraph of G containing all vertices of G. By a decomposition of a graph G into factors we mean such a system $\mathscr{S}$ of factors of the graph G that every edge of G is contained in exactly one factor of $\mathscr{S}$. The diameter $d(G)$ of a directed graph G is the supremum of the set of all distances $\varrho_G(x, y)$ between the pairs of vertices (x, y) of G. The diameter $d(G)$ can be also equal to ∞, if G is not strongly connected or if there does not exist the maximum of the distances between pairs of vertices of G, which may occur in infinite graphs. The other terms are used in the usual sense [3].

Denote by $E(d_1, d_2)$ the smallest cardinal number n such that the graph $\langle\langle n \rangle\rangle$ can be decomposed into two factors with the diameters d_1 and d_2. If such a cardinal number does not exist, we shall write $E(d_1, d_2) = \infty$.

Theorem 1.

$$E(d_1, d_2) = \begin{cases} d_2 + 1 & \text{if } 2 \leq d_1 \leq d_2 < \infty, \\ \infty & \text{if } 1 = d_1 \leq d_2 < \infty, \\ d_1 + 1 & \text{if } 1 \leqq d_1 < d_2 = \infty, \\ 2 & \text{if } d_1 = d_2 = \infty. \end{cases}$$

Proof. Denote the vertices of the graph $\langle\langle n \rangle\rangle$ by symbols v_i for $i = 0, 1, 2, \ldots, n - 1$. To prove the first relation it is sufficient to decompose the graph

$\langle\langle d_2 + 1\rangle\rangle$ into two factors such that the factor F_2 with diameter d_2 contains the edges (Fig. 1):

(1) $v_i v_{i-1}$ for $i = 1, 2, \ldots, d_2 - 1, d_2$,
(2) $v_k v_i$ for $k = 0, 1, \ldots, d_1 - 3$, $i = 2, 3, \ldots, d_2$ and $i - k \geq 2$,
(3) $v_{d_1-2} v_{d_2}$.

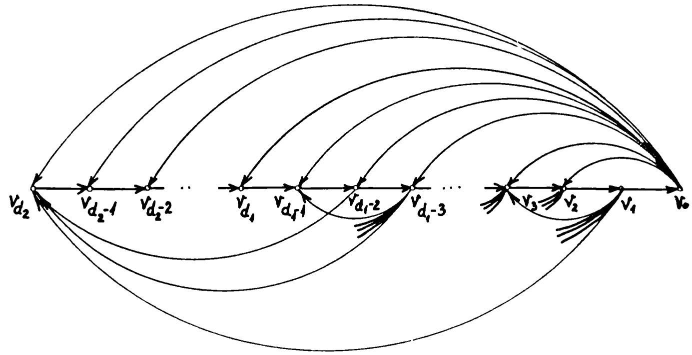

Fig. 1.

The factor F_1 with the diameter d_1 is complementary to F_2 in $\langle\langle d_2 + 1\rangle\rangle$. The distance between two arbitrary vertices of F_2 is less than or equal to the diameter d_2 (because all the vertices of F_2 are on a cycle(1) of the length $d_2 + 1$) and the maximal distance d_2 is attained between the vertices v_{d_2} and v_0. We shall decompose the set V of the vertices F_1 into two sets:

$$V_1 = \{v_0, v_1, \ldots, v_{d_1-2}\} \quad \text{and} \quad V_2 = \{v_{d_1-1}, v_{d_1}, \ldots, v_{d_2-1}, v_{d_2}\}.$$

All the vertices of V_1 and two arbitrary vertices of V_2 are on a cycle of the length $d_1 + 1$, which implies that the distance between two arbitrary vertices of V in F_1 is less than or equal to d_1. The distance d_1 is attained between v_0 and v_{d_2}.

The second relation is evident.

To prove the third relation we construct a decomposition of $\langle\langle d_1 + 1\rangle\rangle$ into two factors F_1 and F_2; F_1 with the diameter d_1 consists of the following edges (Fig. 2):

(1) $\qquad v_i v_{i+1}$ for $i = 0, 1, \ldots, d_1 - 1$,
(2) $\qquad v_{d_1} v_i$ for $i = 0, 1, \ldots, d_1 - 1$.

The diameter of F_1 is d_1. The maximal distance is attained between the vertices v_0 and v_{d_1}. The distance between two arbitrary vertices of F_1 is finite and less

(1) — directed circuit [3].

than or equal to the diameter d_1, because all the vertices of F_1 are on a cycle with the length $d_1 + 1$. The factor F_2 with the diameter d_2 is complementary to F_1 in $\langle\langle d_1 + 1 \rangle\rangle$ and, according to (2), F_2 has the diameter equal to ∞ (because F_2 has not any edge of the type $v_{d_1}v_i$ for $i = 0, 1, \ldots, d_1 - 1$).

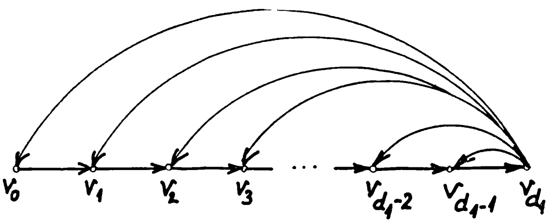

Fig. 2.

To prove the statement $E(\infty, \infty) = 2$, it is sufficient to decompose the graph $\langle\langle 2 \rangle\rangle$ into two factors in the following way: F_1 contains the edge v_0v_1 and F_2 contains the edge v_1v_0.

From Theorem 1 of 2] the following corollary follows:

Corollary 1. *The graph $\langle\langle n \rangle\rangle$ is decomposable into two factors with diameters d_1 and d_2 if and only if $n \geq E(d_1, d_2)$, where $E(d_1, d_2)$ is the same as in Theorem* 1:

For the case of three factors our results are not complete. Namely, we have.

Theorem 2.

$$E(d_1, d_2, d_3) = \begin{cases} 2 & \text{if } d_1 = d_2 = d_3 = \infty, \\ d_1 + 1 & \text{if } 1 \leq d_1 < d_2 = d_3 = \infty, \\ d_2 + 1 & \text{if } 2 \leq d_1 \leq d_2 < d_3 = \infty, \\ \infty & \text{if } 1 = d_1 \leq d_2 \leq d_3, \quad d_2 < \infty, \\ d_3 + 1 & \text{if } 2 = d_1 \leq d_2 \leq d_3 < \infty \\ & \quad \text{and } d_1 + d_2 + d_3 \geq 10. \end{cases}$$

Proof. The first three relations follow from Theorem 1. The fourth relation is obvious. The fifth relation follows from Theorem 7 of [1] and from decompositions given in Table 1. (Edges joining vertices v_i and v_j we denote ij.)

There exist also decompositions of $\langle\langle 6 \rangle\rangle$ into three factors with diameters $d_1 - 2$, $d_2 = d_3 = 4$, of $\langle\langle 7 \rangle\rangle$ into three factors with diameters $d_1 = 2$, $d_2 = 3$, $d_3 = 5$ and of $\langle\langle 8 \rangle\rangle$ into three factors with diameters $d_1 = d_2 = 2$, $d_3 = 6$. (See Table 1.) From the existence of these decompositions, from Theorem 2 (of this paper) and from Theorems 1 and 7 of [1] we have:

Corollary 2. *Let three diameters d_1, d_2, d_3 be given. Let one of the following cases occur*:

I. *One of the diameters is* ∞;

Table 1

Number of vertices	Edges of factors			Diameters		
	F_1	F_2	F_3	d_1	d_2	d_3
5	13, 31, 14, 41, 24, 42, 25, 52, 35, 53	12, 51, 23, 34, 45	21, 15, 32, 43, 54	2	4	4
6	13, 31, 14, 41, 24, 42, 25, 52, 26, 62, 35, 53, 36, 63	12, 15, 51, 61, 23, 34, 45, 56	21, 16, 32, 43, 54, 46, 64, 65	2	4	4
6	13, 31, 14, 41, 24, 42, 25, 52, 26, 62, 35, 53, 36, 63, 46, 64	12, 15, 51, 61, 23, 34, 45, 56	21, 16. 32, 43, 54, 65	2	4	5
6	13, 31, 14, 41, 15, 51, 24, 42, 25, 52, 26, 62, 35, 53, 36, 63, 46, 64	12, 61, 23, 34, 45, 56	21, 16, 32, 43, 54, 65	2	5	5
6	12, 31, 14, 51, 61, 23, 42, 25, 62, 34, 53, 45, 46, 56	13, 41, 15, 24, 52, 26, 35, 36, 63, 64	21, 16, 32, 43, 54, 65	2	3	5
7	12, 31, 14, 51, 61, 23, 42, 25, 62, 27, 34, 53, 37, 73, 45, 46, 74, 56	13, 41, 16, 71, 24, 26, 72, 35, 36, 63, 64, 47, 57, 75, 67	21, 15, 17, 32, 52, 43, 54, 65, 76	2	3	5
7	12, 31, 14, 51, 61, 23, 42, 25, 62, 27, 34, 53, 37, 73, 45, 46, 74, 56	13, 41, 15, 16, 71, 24, 52, 26, 72, 35, 36, 63, 64, 47, 57, 67, 75	21, 17, 32, 43, 54, 65, 76	2	3	6
7	12, 31, 14, 61, 23, 42, 25, 62, 27, 34, 53, 37, 73, 45, 46, 74, 56	13, 41, 15, 51, 16, 71, 24, 52, 26, 72, 35, 36, 63, 64, 47, 65, 57	21, 17, 32, 43, 54, 75, 67, 76	2	2	6
8	12, 31, 14, 61, 23, 42, 25, 62, 27, 28, 34, 53, 37, 73, 38, 83, 45, 46, 74, 84, 56	13, 41, 15, 51, 16, 17, 71, 81, 24, 52, 26, 72, 82, 35, 36, 63, 64, 47, 48, 65, 75, 58, 67, 76, 87	21, 18, 32, 43, 54, 57, 85, 68, 86, 78	2	2	6
8	12, 31, 14, 61, 23, 42, 25, 62, 27, 28, 34, 53, 37, 73, 38, 83, 45, 46, 74, 84, 56	13, 41, 15, 51, 16, 71, 81, 24, 52, 26, 17, 72, 82, 35, 36, 63, 64, 47, 48, 65, 57, 58, 85, 76, 68, 78, 87	21, 18, 32, 43, 54, 75, 67, 86	2	2	7

II. *One of the diameters is* 1,

III. *One of the diameters is* 2 *and* $d_1 + d_2 + d_3 \geq 10$.

Then $\langle\langle n \rangle\rangle$ *is decomposable into three factors with diameters* d_1, d_2, d_3 *if and only if* $n \geq E(d_1, d_2, d_3)$.

Remark. We do not know whether the assertion of Corollary 2 holds generally, for arbitrary three given diameters. From [2] we know that the analogical assertion for the case of four factors does not hold in general.

REFERENCES

[1] Bosák J., Rosa A., Znám Š., *On decompositions of complete graphs into factors with given diameters*, Theory of graphs, Proceedings of the colloquium held at Tihany, Hungary, Sept. 1966, Akadémiai Kiadó, Budapest 1968, 37—56.

[2] Znám Š., *Decomposition of the complete directed graph into two factors with given diameters*, Mat. časop. *20* (1970), 254—256.

[3] Ore O., *Theory of graphs*, Amer. Math. Soc., Providence 1962.

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Matematický ústav
Slovenskej akadémie vied
Bratislava