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PROPERTIES OF THE NONOSCILLATORY SOLUTION FOR A THIRD ORDER NONLINEAR DIFFERENTIAL EQUATION

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J. L. Nilson proved in paper [1] a stability theorem for a solution of a nonlinear differential equation

$$x'''(t) + p(t)x'(t) + q(t)x^{2n+1}(t) = 0, \quad n = 1, 2, 3, \dots .$$

In this paper we shall prove for the nonlinear differential equation

$$x'''(t) + p(t)x'(t) + q(t)x^{\alpha}(t) = 0,$$

where $\alpha > 1$, $\alpha = p/q$, p and q are nondivisible odd integers, a theorem similar to that proved in paper [1].

The solution $x(t)$ of a differential equation

$$(1) \qquad x'''(t) + p(t)x'(t) + q(t)x^{\alpha}(t) = 0, \quad \alpha \geqq 0$$

is called nonoscillatory if there exists in the interval (a, ∞), $a > 0$ such a number t_0, that for all $t > t_0$ $x(t) \neq 0$ is valid.

In what follows, the following lemma will be useful.

Lemma. *Let $\mathbf{f} = \{f_1, \dots, f_n\}$ and $\mathbf{y}$ be vectors. Let the function $\mathbf{f}(t, \mathbf{y})$ be continuous for $t \geqq 0$, $\mathbf{y} \geqq 0$. The symbol $\mathbf{y} \geqq 0$ means that $\mathbf{y} = \{y_1, \dots, y_n\}$ and $y_i \geqq 0, i = 1, 2, \dots, n$. Let the function $\mathbf{f}(t, \mathbf{y})$ be such that $\mathbf{f}(t, \mathbf{0}) = \mathbf{0}$, $\mathbf{f}(t, \mathbf{y}) \geqq 0$ for $\mathbf{y} \geqq 0$. Let c be an arbitrary nonnegative number. Then the differential equation $\mathbf{y}' = -\mathbf{f}(t, \mathbf{y})$ has at least one solution $\mathbf{y}(t)$ defined for $t \geqq 0$ such that $\|\mathbf{y}(0)\| = 0$ and $\mathbf{y}(t) \geqq 0$, $\mathbf{y}'(t) \leqq 0$.*

Proof. See [2] p. 510; 2,8.

Theorem 1. *Let the functions $p(t)$, $q(t)$ be continuous. Let $q(t) \geqq 0$, $p(t) < 0$ for large t. Then there exists at least one nonoscillatory solution $x(t)$ of the equation (1) such that for large t $x(t) \geqq 0$, $x'(t) \leqq 0$ holds.*

Proof. Putting $x(t) = y_1(t)$, $x'(t) = -y_2(t)$, $x'''(t) = y_3(t)$, the differential equation (1) can be written as the following system of the differential equations

$$(2)\qquad \begin{aligned} y_1'(t) &= -y_2(t)\\ y_2'(t) &= -y_3(t)\\ y_3'(t) &= -[q(t)y_1^\alpha(t) - p(t)y_2(t)], \end{aligned}$$

or briefly

$$\mathbf{y}' = -\mathbf{f}(t, y),$$

where

$$\mathbf{y} = \{y_1(t), y_2(t), y_3(t)\}, \quad \mathbf{y}' = \{y_1'(t), y_2'(t), y_3'(t)\}$$

and

$$\mathbf{f}(t, \mathbf{y}) = \{y_2(t), y_3(t), [q(t)y_1^\alpha(t) - p(t)y_2(t)]\}.$$

It can be easily verified that all the assumptions of the Lemma are fulfilled. Thus, according to the Lemma the differential equation (1) has at least one nonoscillatory solution $x(t)$ such that $x(t) \geqq 0$, $x'(t) \leqq 0$ for large t.

Theorem 2. *Let $\alpha > 1$, $\alpha = p/q$, where p and q are nondivisible odd natural numbers. Let the functions $p(t)$ and $q(t)$ satisfy the following conditions for the large t:*

1) *$q(t)$ is nonnegative and continuous;*
2) *$p(t)$, $p'(t)$ are continuous and $p(t) < 0$, $p'(t) \geqq 0$;*
3) *for any constants A, B and for the large t we have $A + Bt - \int_{t_\cdot}^{t} Q(s)\,\mathrm{d}s < 0$, where $Q(t) = \int_{t_0}^{t} q(s)\,\mathrm{d}s$.*

Then any nonoscillatory solution $x(t)$ of the nonlinear differential equation

$$(3)\qquad x'''(t) + p(t)x'(t) + q(t)x^\alpha(t) = 0$$

has the following properties for large t:

a) $\operatorname{sgn} x(t) = \operatorname{sgn} x''(t) \neq \operatorname{sgn} x'(t)$, *where*

$$\operatorname{sgn} x(t) = \begin{cases} 1 \ \textit{if}\ x(t) \geqq 0, \\ -1 \ \textit{if}\ x(t) < 0; \end{cases}$$

b) $\lim_{t\to\infty} x''(t) = \lim_{t\to\infty} x'(t) = 0, \quad \lim_{t\to\infty} |x(t)| = L \geqq 0;$

c) *$x(t)$, $x'(t)$, $x''(t)$ are monotone functions.*

Proof. From Theorem 1 it follows that the differential equation (3) has a nonoscillatory solution. Let $x(t)$ be any nonoscillatory solution of the differential equation (3). Let t_0 be a large positive number such that $x(t) \neq 0$ for all $t > t_0$. Since $-x(t)$ is also a solution of the differential equation (3), without loss of generality, assume that $x(t) > 0$ for all $t > t_0$. The differential equation (3) can be written in the form

$$(4) \qquad \frac{x'''(t)}{x^{\alpha}(t)} + \frac{p(t)x'(t)}{x^{\alpha}(t)} = -q(t) \text{ for } t \geqq t_0 .$$

An integration from t_0 to t, an integration by parts gives

$$(5) \qquad \frac{x''(t)}{x^{\alpha}(t)} + \frac{\alpha}{2}\frac{x'^2(t)}{x^{\alpha+1}(t)} + \frac{\alpha(\alpha+1)}{2}\int_{t_0}^{t}\frac{x'^3(s)}{x^{\alpha+2}(s)}\,\mathrm{d}s -$$

$$-\frac{1}{\alpha-1}\frac{p(t)}{x^{\alpha-1}(s)} + \frac{1}{\alpha-1}\int_{t_0}^{t}\frac{p'(s)}{x^{\alpha-1}(s)}\,\mathrm{d}s = K - \int_{t_0}^{t} q(s)\,\mathrm{d}s .$$

An integration from t_0 to t of the equality (5) gives

$$(6) \qquad \frac{x'(t)}{x^{\alpha}(t)} + \frac{3\alpha}{2}\int_{t_0}^{t}\frac{x'^2(s)}{x^{\alpha+1}(s)}\,\mathrm{d}s + \frac{\alpha(\alpha+1)}{2}\int_{t_0}^{t}\frac{(t-s)x'^3(s)}{x^{\alpha+2}(s)}\,\mathrm{d}s -$$

$$-\frac{1}{\alpha-1}\int_{t_0}^{t}\frac{p(s)}{x^{\alpha-1}(s)}\,\mathrm{d}s + \frac{1}{\alpha-1}\int_{t_0}^{t}\frac{(t-s)p'^3(s)}{x^{\alpha-1}(s)}\,\mathrm{d}s = M + Kt - \int_{t_0}^{t} Q(s)\,\mathrm{d}s,$$

where $Q(s) = \int_{t_0}^{t} q(s)\,\mathrm{d}s$.

At first it will be proved that for an arbitrary $t_0' > t_0$ the function $x'(t)$ cannot be nonnegative for all $t > t_0'$. Suppose that $x'(t) \geqq 0$ for all $t > t_0'$. Let t_p be such a chosen number that the conditions of Theorem 2 hold for all $t \geqq t_p$ and $t_p \geqq t_0'$. For $t \geqq t_p$ the following holds:

$$(7) \qquad \frac{x'(t)}{x^{\alpha}(t)} + \frac{\alpha(\alpha+1)}{2}\int_{t_p}^{t}\frac{(t-s)x'^3(s)}{x^{\alpha+2}(s)}\,\mathrm{d}s - \frac{1}{\alpha-1}\int_{t_p}^{t}\frac{p(s)}{x^{\alpha-1}(s)}\,\mathrm{d}s +$$

$$+\frac{1}{\alpha-1}\int_{t_p}^{t}\frac{(t-s)p(s)}{x^{\alpha-1}(s)}\,\mathrm{d}s \leqq \bar{M} + Kt - \int_{t_p}^{t} Q(s)\,\mathrm{d}s,$$

where all the constants are combined and called $\bar{M}$. For $t \geqq t_p$ the right-hand side $M + Kt - \int_{t_p}^{t} Q(s)\,\mathrm{d}s$ is negative and the left-hand side of the equation (7) positive. This is clearly impossible. There are two possibilities for $x'(t)$:

a) There exists $\bar{t}$ such that $x'(t) < 0$ for $t > \bar{t}$;

b) for each $t \in (t_0, \infty)$ there exists $\bar{t} > t$ such that $x'(\bar{t}) \geqq 0$.

The case b) is not possible. In fact, let $t_1 > t$ such that $x'(t_1) \geqq 0$. There exists a number $t_2 > t_1$ such that $x'(t_2) < 0$. Let r be the greatest zero of $x'(t)$ less than t_2. There exists a number $t_3 > t_2$ such that $x'(t_3) \geqq 0$. Let s be the smallest zero of $x'(t)$ greater than t_2. Multiplying the differential equation (3) by $x'(t)$, we obtain

$$x'''(t)x'(t) + p(t)x'^2(t) + q(t)x^\alpha(t)x'(t) = 0. \tag{8}$$

Integrating from r to s the equality (8), we have

$$-\int_r^s x''^2(t)\,dt + \int_r^s p(t)'^2(t)\,dt + \int_r^s q(t)x^\alpha(t)x'(t)\,dt = 0. \tag{9}$$

But the left-hand side of equality (9) is negative for the large t, which is impossible. Hence there exists a $\bar{t}$ such that $x'(t) < 0$ for all $t > \bar{t}$.

In what follows $\lim_{t\to\infty} x''(t) = \lim_{t\to\infty} x'(t) = 0$ will be proved. Let us write the differential equation (3) in the form

$$x'''(t) = -p(t)x'(t) - q(t)x^\alpha(t),$$

the right-hand side is negative for large t. Therefore $x'''(t) < 0$ for all $t > \bar{t}$. This implies that $x''(t)$ is a decreasing function and $x'(t)$ is concave downward for $t > \bar{t}$. There are three possibilities for $x'(t)$:

1. $\lim_{t\to\infty} x'(t) = -\infty$;

2. $\lim_{t\to\infty} x'(t) = c < 0$;

3. $\lim_{t\to\infty} x'(t) = 0$.

Case 1 is impossible since it implies that $x(t)$ is negative for large t, which is a contradiction with the assumption. From case 2 it follows that $x(t)$ is negative for large t, which is a contradiction with the assumption. Therefore, the only possibility remaining is $\lim_{t\to\infty} x'(t) = 0$.

Since $x''(t)$ is decreasing it must be positive for large t, otherwise $\lim_{t\to\infty} x'(t) = -\infty$, hence $x'(t)$ is monotone increasing. Since $x''(t)$ is monotone decreasing and positive, $\lim_{t\to\infty} x''(t)$ exists. We shall prove that $\lim_{t\to\infty} x''(t) = 0$. Suppose that $\lim_{t\to\infty} x''(t) = c > 0$. Then $x'(t) > ct + k > 0$ for large t, this is impossible since $x'(t) < 0$ for large t. Therefore, $\lim_{t\to\infty} x''(t) = 0$. Thus $x(t)$ is positive decreasing and concave upward for large t.

Corollary. *If the assumptions of Theorem* 2 *are fulfilled and* $0 < \varepsilon < q(t)$ *for large* t, *then for the nonoscillatory solution* $x(t)$ *of the differential equation* (3) $\lim_{t\to\infty} x(t) = 0$ *holds.*

Proof. Suppose $\lim_{t\to\infty} x(t) = L$, $L \neq 0$. Since $-x(t)$ is a solution whenewer $x(t)$ is a solution, it can be assumed without loss of generality that $L > 0$. Then for large t the inequality $0 < L < x(t)$ holds.

The last inequality gives for large t

$$-\varepsilon x^{\alpha}(t) < -\varepsilon L^{\alpha} < 0 .$$

From the assumption $0 < \varepsilon < q(t)$ for large t it follows

$$-q(t)x^{\alpha}(t) < -\varepsilon x^{\alpha}(t) .$$

Further for large t $p(t)x'(t) > 0$ holds. Thus for large t

$$x'''(t) = -p(t)x'(t) - q(t)x^{\alpha}(t) < -q(t)x^{\alpha}(t) < -\varepsilon x^{\alpha}(t) < -\varepsilon L^{\alpha} < 0$$

and $\lim_{t\to\infty} x''(t) = -\infty$, which is impossible, because $\lim_{t\to\infty} x''(t) = 0$. Hence $L = 0$ and $\lim_{t\to\infty} x(t) = 0$.

The following example illustrates Theorem 2. We consider the differential equation

$$x'''(t) - \frac{1}{2}\, x'(t) + \frac{1}{2}\, e^{2t/3} x^{5/3}(t) = 0 .$$

The function $x(t) = e^{-t}$ is a solution with the required properties.

Remark. Theorem 2 does not hold for $\alpha = 1$, i.e. in the linear case. We consider the differential equation

$$x'''(t) - 2x'(t) + x(t) = 0 .$$

Its nonoscillatory solution $x(t) = e^{t}$ has not the properties required by Theorem 2.

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