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# ON RADICALS IN A CERTAIN CLASS OF SEMIGROUPS

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Let $S$ be a semigroup and consider the law $xyzyx = yxzxy$ for all $x, y, z \in S$. The commutative semigroups and all subsemigroups of class 2 nilpotent groups (see Neumann-Taylor [3]) satisfy this law. Accordingly, this law will be called the $C_2$-law and any semigroup which satisfies it will be referred to as a $C_2$-semigroup. J. Bosák [1, p. 209] and R. Šulka [4, p. 221] proved that if $S$ is a commutative semigroup and $J$ is an ideal of $S$, then the Clifford, McCoy, Ševrin, and completely prime radicals with respect to $J$ ($R^*(J)$, $M(J)$, $L(J)$, $C(J)$) are equal to $N(J)$, the set of all nilpotent elements with respect to $J$. The reader is referred to the papers by J. Bosák and R. Šulka for relevant definitions.

The purpose of this note is to extend the above mentioned result to the class of $C_2$-semigroups obtaining the following:

**Theorem.** *If $S$ is a $C_2$-semigroup and $J$ is an ideal of $S$, then $M(J) = L(J) = R^*(J) = C(J) = N(J)$.*

Proof. It will first be shown that $M(J) = L(J) = R^*(J) = N(J)$.

(1) Let $S$ be a $C_2$-semigroup and $M$ an $m$-system in $S$. Then, for any $r \in M$ and any positive integer $n$, there exists $s \in S$ such that $r^{2^n}sr^{2^n} \in M$.

This follows using induction on the integer $n$. Since $r \in M$, there exist $x, y \in S$ so that $rxr \in M$ and $rxryrxr \in M$. But by the $C_2$-law $rxryrxr = r^2xyxr^2 \in M$. Thus the statement is true for $n = 1$. Assume the statement is true for $n = k$. Then, by the induction hypothesis, there is a $z \in S$ so that $r^{2^k}zr^{2^k} \in M$. Since $M$ is an $m$-system, there exists $t \in S$ so that $r^{2^k}zr^{2^k}tr^{2^k}zr^{2^k} \in M$. Using the $C_2$-property of $S$, $r^{2^k}zr^{2^k}tr^{2^k}zr^{2^k} = r^{2^k} . r^{2^k}ztzr^{2^k} . r^{2^k} = r^{2^{k+1}}ztzr^{2^{k+1}}$ and the case for $n = k + 1$ is proved. Hence the statement is valid for any positive integer $n$.

(2) It may now be shown that $N(J) \subseteq M(J)$. For, let $r \in N(J)$ and $M$ be an $m$-system containing $r$. There exist a natural number $k$ so that $r^k \in J$ and also a natural number $n$ such that $2^n \geq k$. By (1), it is possible to find an element $t \in S$ so that $r^{2^n}tr^{2^n} \in M$. Thus, any $m$-system containing $r$ meets $J$ and the statement is proved.

(3) $M(J) = L(J) = R^*(J) = N(J)$. This is true as a consequence of (2) and

a theorem by J. Bosák [1, p. 209] stating that $M(J) \subseteq L(J) \subseteq R^*(J) \subseteq N(J)$ for an ideal $J$ in an arbitrary semigroup $S$.

The proof of the theorem will be completed by showing that $N(J) = C(J)$. McCoy [2], though working in commutative rings, provides a proof easily adapted for abelian semigroups. In this work, part (4), a key result in McCoy's approach, is proved to be valid for $C_2$-semigroups. The remainder of the proof follows as for the commutative case and is presented here for completeness.

(4) Let $M$ be a subsemigroup of a $C_2$-semigroup and $J$ an ideal which does not meet $M$. Then $J$ is contained in a maximal ideal $P$ which does not meet $M$; and $P$ is completely prime.

The proof is as follows.

Let $P$ be the union of all ideals which contain $J$ but do not meet $M$. The ideal $P$ has the required maximal property and it remains to show that $P$ is completely prime.

Let us assume that $a \notin P$ and $b \notin P$ and show that $ab \notin P$. Since $a \notin P$, it is clear that the ideal generated by $P$ and $a$, $i(P, a)$, contains $P$ properly. Because of the maximal property of $P$, this implies that $i(P, a)$ contains an element $m_1$ of $M$. Similarly, $i(P, b)$, contains an element $m_2$ of $M$.

Now, $m_1$ may have several forms: $a, sa, as$, or $sat$, where $s, t \in S$.

Similarly, $m_2$ may have several forms: $b, s_1b, bs_1$, or $s_1bt_1$, where $s_1, t_1 \in S$.

If $m_1 = a$ and $m_2 = b$, it is easy to see that $ab \notin P$, for the contrary would imply that $m_1m_2 \in P$. If $m_1 = sa$ and $m_2 = b$ or $bs_1$, it is obvious that the situation is analogous to the above. Should $m_1 = sa$ and $m_2 = s_1b$, then by the $C_2$-law, $m_1m_2^3 = sa(s_1b)^3 = sa \,.\, bs_1^3b^2$ which implies that $ab$ cannot belong to $P$. If $m_1 = as$ and $m_2$ is of the form $b, s_1b$ or $bs_1$, then arguments similar to those just presented may be used. Now, consider $m_1 = sat$ and $m_2 = s_1bt_1$. Then using the $C_2$-law again, $m_1^3 = sa \,.\, tsatsat = (sa)^{2}t^3sa$ and $m_2^3 = s_1bt_1s_1bt_1s_1 \,.\, bt_1 = bt_1s_1^3(bt_1)^2$ so that $m_1^3m_2^3$ implies that $ab$ cannot belong to $P$.

The remaining cases:

(i) $m_1 = a$ and $m_2 = s_1b,\ bs_1$ or $s_1bt_1$,

(ii) $m_1 = sa$ and $m_2 = s_1bt_1$,

(iii) $m_1 = as$ and $m_2 = s_1bt_1$,

(iv) $m_1 = sat$ and $m_2 = b,\ s_1b$ or $bs_1$,

reduce to arguments similar to the above.

(5) Following McCoy [2, p. 106], in conjunction with (4), it may be demonstrated that: A set $P$ of elements in a $C_2$-semigroup $S$ is a minimal completely prime ideal belonging to the ideal $J$ if and only if the set complement of $P$ with respect to $S$, $S - P$, is a maximal subsemigroup which does not meet $J$.

(6) $N(J) = C(J)$.

Clearly, any completely prime ideal which contains $J$ also contains $N(J)$. Thus, $N(J)$ is contained in the intersection of all minimal completely prime ideals belonging to $J$. The proof will be completed by showing that if $a$ is an element of $S$ not in $N(J)$, then there is a minimal completely prime ideal belonging to $J$ which does not contain $a$. If $a$ is such an element, then consider the set $M$ of all elements of the form $a^i$, $i = 1, 2, \ldots$ . $M$ is a subsemigroup which does not meet $J$ and, by a standard application of Zorn's Lemma, is contained in a maximal subsemigroup $M'$ which does not meet $J$. Since $a \in M'$, $a \notin S - M'$ which, by (5), is a minimal completely prime ideal which belongs to $J$. Hence, $a$ is not in the intersection of all the minimal completely prime ideals belonging to $J$, and this concludes the proof.

The main result follows as a consequence of parts (3) and (6).

## REFERENCES

[1] Bosák J., *On Radicals of Semigroups*, Mat. časop. *18* (1968), 204—212.

[2] McCoy N. H., *Rings and Ideals*, Carus Monographs, Vol. *8*, Mathematical Association of America, Buffalo, New York, 1948.

[3] Neumann B. H., Taylor T., *Subsemigroups of Nilpotent Groups*, Proc. Roy. Soc. Ser. A, *274* (1963), 1—4.

[4] Šulka R., *On Nilpotent Elements, Ideals, and Radicals of a Semigroup*, Mat.-fyz. časop. *13* (1963), 209—222.

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## ERRATA

Šalát T. and Znám Š., *Correction to our paper „On the average order of an arithmetical function"*, Mat. časop. *20* (1920), 233—238.

The correct formulation of the Theorem (p. 233) is as follows: *Theorem.*

$$\lim_{N\to\infty} \frac{1}{N} [f(1) + f(2) + \ldots + f(N)] \frac{\log N}{N} = \frac{\pi^2}{12}.$$