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ON AN ABSTRACT FORMULATION OF ABSOLUTE CONTINUITY AND DOMINANCY

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Some theorems, the proofs of which depend on the absolute continuity of measures, have been proved in [2], without the means of measure theory. There, the absolute continuity formulated in terms of the sets of zero-measure has been considered. Analogical results, without the means of measure theory may be obtained for the $\varepsilon - \delta$ absolute continuity ([1] p. 97). To develop such results is the aim of this paper. The abstract formulation given here is based on axiomatization of systems of sets having „small measure". Some of the axioms for the mentioned systems are modifications of those given by B. Riečan in [3]. In what follows $(X,\mathscr{S})$ denotes a measurable space in the sense of [1].

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Given $\mathscr{I} \subset \mathscr{S}$ and $\mathscr{I}^* \subset \mathscr{S}$ such that $\mathscr{I}$ and $\mathscr{I}^*$ are σ-ideals, the symbol $\mathscr{I} < \mathscr{I}^*$ means that $\mathscr{I}^* \subset \mathscr{I}$. In this case we shall say ,as in [1], that $\mathscr{I}$ is absolutely continuous with respect to $\mathscr{I}^*$. To simplify the notation, we use $<$ (see also [5]) instead of the more usual $\ll$ In what follows systems $\mathscr{N}_n$ will be supposed to satisfy certain axioms. These axioms will be introduced subsequently as they will be used in the proofs. Their relation to the axioms in [3] will be discussed at the end of the paper.

Given $\{\mathscr{N}_n\}$, the symbol $\mathscr{N}$ denotes $\bigcap_{n=1}^{\infty} \mathscr{N}_n$. In case the systems depend on a parameter t belonging to a given set T, $\mathscr{N}^t$ denotes $\bigcap_{n=1}^{\infty} \mathscr{N}_n^t$. A mapping τ of X into X will be called a transformation. In what follows τ will be supposed to be measurable. Moreover in some of the theorems it will be supposed to satisfy some of the following three properties: α) τ is nonsingular with respect to $\mathscr{E} \subset \mathscr{S}$ (i. e. if $E \in \mathscr{E}$ then $\tau^{-1}(E) \in \mathscr{E}$). β) if $E \notin \mathscr{K}$ ($\mathscr{K} \subset \mathscr{S}$), then $\bigcap_{k=1}^{\infty} \tau^{-k}(E) \notin$

$\notin \mathscr{K}$. γ) For any sequence $\{E_k\}_{k-1}^{\infty}$ of pairwise disjoint sets $E_k \in \mathscr{L}$ ($\mathscr{L} \subset \mathscr{S}$), the following is supposed to hold: $\bigcap_{k=1}^{\infty} \tau^{-k} (\bigcup_{s=1}^{\infty} E_s) = \bigcup_{s=1}^{\infty} \bigcap_{k=1}^{\infty} \tau^{-k}(E_s)$.

The first axioms we shall need are the following three:

a) $\mathscr{N}_n \neq \emptyset$; $n = 1, 2, 3, \ldots$

b) For any n the following holds: $E \in \mathscr{N}_n$, $F \in \mathscr{S}$, $F \subset E$ *implies* $F \in \mathscr{N}_n$.

c) For any $\mathscr{N}_n$, $(n = 1, 2, 3 \ldots)$, *there exists a sequence* $\{k_i\}$ *of positive integers such that if* N *is an arbitrary positive integer then there is* $r(N)$ *such that if* $E_i \in \in \mathscr{N}_{ki}$ $(i = 1, 2, 3, \ldots)$, *then* $\bigcup_{i>r(N)} E_i \in \mathscr{N}_N$.

Lemma 1. *For any* $\{\mathscr{N}_n\}$ *satisfying a), b), c), the system* $\mathscr{N} = \bigcap_{n=1}^{\infty} \mathscr{N}_n$ *is a σ-ideal in* $\mathscr{S}$.

Proof. $\mathscr{N}$ is non-empty as it follows immediately from *a)* and *b)*. If $E \in \mathscr{N}$, $F \subset E$, $F \in \mathscr{S}$, then $F \in \mathscr{N}$ by *b)*. Now let $E_r \in \mathscr{N}$ $(r = 1, 2, 3, \ldots)$. Choose $\{k_i\}$ according to *c)*. Then there is $r(N)$ such that $\bigcup_{i>r(N)} F_i \in \mathscr{N}_N$ for any $F_i \in \mathscr{N}_{ki}$. Putting $F_i = \emptyset$ for $i = 1, 2, 3, \ldots$ $r(N)$, $F_{r(N)+1} = E_1$, $F_{r(N)+2} = E_2, \ldots$, we have

$$\bigcup_{r=1}^{\infty} E_i = \bigcup_{i>r(N)} F_r \in \mathscr{N}_N \tag{1}$$

Since N is arbitrary, (1) gives $\bigcup_{n=1}^{\infty} E_i \in \mathscr{N}$.

Definition 1. *Given* $\{\mathscr{N}_n\}$ *and* $\{\mathscr{N}_n^*\}$ *satisfying a), b), c), the system* $\{\mathscr{N}_n\}$ *is said to be absolutely continuous with respect to* $\{\mathscr{N}_n^*\}$ *(notation* $\{\mathscr{N}_n\} < \{\mathscr{N}_n^*\}$*) if and only if* $\mathscr{N} < \mathscr{N}^*$.

Another notion of absolute continuity (this one which will be an analogy of $\varepsilon - \delta$ absolute continuity in measure theory) may be formulated as follows.

Definition 2. *The system* $\{\mathscr{N}_n\}$ *is said to be strongly absolutely continuous with respect to* $\{\mathscr{N}_n^*\}$ *(notation* $\{\mathscr{N}_n\}$*)* $(<)$ $\{\mathscr{N}_n^*\}$ *if to any* n_0 *there exists an* n^* *such that* $E \in \mathscr{N}_n^*$ *implies* $E \in \mathscr{N}_{n_0}$.

Note 1. If μ, μ^* are two measures on $\mathscr{S}$, then putting $\mathscr{N}_n = \left\{E : E \in \mathscr{S}, \mu(E) < \frac{1}{n}\right\}$ and $\mathscr{N}_n^* = \left\{E : E \in \mathscr{S}, \mu^*(E) < \frac{1}{n}\right\}$, we have: μ is absolutely continuous with respect to μ^* in the sense $\varepsilon - \delta$ if and only if $\{\mathscr{N}_n\}$ $(<)$ $\{\mathscr{N}_n^*\}$.

Note 2. Evidently if $\{\mathcal{N}_n\}\ (<)\ \{\mathcal{N}_n^*\}$ then $\{\mathcal{N}_n\} < \{\mathcal{N}_n^*\}$. The converse is in general not true as it is well known ([1], p. 128, 12).

In what follows the notion of the strong absolute continuity of $\{\mathcal{N}_n^t\}$ $(t \in T)$ with respect to $\{\mathcal{N}_n^s\}$ $(s \in S)$, where T, S are given sets of indexes, will be introduced. More generally, the notion of τ-asymptotic strong absolute continuity, where τ is a measurable transformation, will be considered. In any case the last notion will include the strong absolute contunuity which will be obtained when τ will be the identical transformation which trivially satisfies the conditions α) β) γ). Let us note that the notion of τ-asymptotic absolute continuity (not that of the strong one) was given for σ-ideals in [2] as follows: If $\{\mathscr{I}^t\}$ $(t \in T)$ and $\{\mathscr{I}^s\}$ $(s \in S)$ are two systems of σ-ideals in $\mathscr{S}$ the $\{\mathscr{I}^t\} <_\tau$ $<_\tau \{\mathscr{I}^s\}$ iff $E \in \bigcap_{s\in S} \mathscr{I}_s$ implies $\bigcap_{k=1}^{\infty} \tau^{-k}(E) \in \bigcap_{t\in T} \mathscr{I}_t$.

Definition 3. *Let T, S be nonempty sets and $\{\mathcal{N}_n^t\}$, $\{\mathcal{N}_n^s\}$ $(t \in T)$, $(s \in S)$ two collections of systems satisfying a), b), c). Let τ be any measurable transformation of X into X. The collection $\{\mathcal{N}_n^t\}$ is said to be strongly τ-asymtoticaly absolutely continuous (or τ-asymptoticaly uniformly continuous) with respect to $\{\mathcal{N}_n^s\}$ if corresponding to any natural number n_0 there exists n' such that $E \in \bigcap_{s\in S} \mathcal{N}_{n'}^s$ implies $\bigcap_{k=1}^{\infty} \tau^{-k}(E) \in \mathcal{N}_{n_0}^t$ for every $t \in T$. (Notation $\{\mathcal{N}_n^t\}\ (<_\tau)\ \{\mathcal{N}_n^s\}$)*

Note 3. If S is an one-point set then $\{\mathcal{N}_n^t\}$ is said to be strongly (or uniformly) τ-dominated. If τ is an identical transformation, then we say that $\{\mathcal{N}_n^t\}$ is strongly (or uniformly) absolutely continuous with respect to $\{\mathcal{N}_n^s\}$ (Notation $\{\mathcal{N}_n^t\}\ (<)\ \{\mathcal{N}_n^s\}$).

Note 4. The notion of τ-dominancy or τ-sasymptotic absolute continuity is introduced by the corresponding notion for the related σ-ideals. (I. e. $\{\mathcal{N}_n^t\}$ is said to be τ-asymptoticaly absolutely continuous with respect to $\{\mathcal{N}_n^s\}$. Notation $\{\mathcal{N}_n^t\} <_\tau \{\mathcal{N}_n^s\}$ iff $\mathcal{N}^t <_\tau \mathcal{N}^s$).

Note 5. Evidently $\{\mathcal{N}_n^t\}\ (<_\tau)\ \{\mathcal{N}_n^s\}$ implies $\{\mathcal{N}_n^t\} <_\tau \{\mathcal{N}_n^s\}$.

Note 6. If $\{\mu_t\}$ $(t \in T)$ is a system of measures and μ is a measure, then $\{\mu_t\}$ is uniformly absolutely continuous with respect to μ, just when the corresponding collection belonging to $\{\mu_t\}$ (see note 1) is uniformly absolutely continuous (uniformly dominated) with respect to the system $\{\mathcal{N}_n\}$ belonging to μ.

Note 7. To any type of the absolute continuity the corresponding type of an equivalency is defined in a natural way. E. g. if $\{\mathcal{N}_n^t\}\ (<_\tau)\ \{\mathcal{N}_n^s\}$ and simultaneously $\{\mathcal{N}_n^s\}\ (<_\tau)\ \{\mathcal{N}_n^t\}$, then $\{\mathcal{N}_n^s\}$ and $\{\mathcal{N}_n^t\}$ are called τ-asymptotically strongly equivalent. $(\{\mathcal{N}_n^s\} \sim_\tau \{\mathcal{N}_n^t\})$.

Definition 4. *If a collection $\{\mathcal{N}_n^t\}$ $(t \in T)$ is given such that c) is satisfied for every $t \in T$ with the same sequence $\{k_i\}$, then the collection $\{\mathcal{N}_n^t\}$ is said to be uniform.*

Note 8. A collection $\{\mathcal{N}_n^t\}$ belonging to any system $\{\mu_t\}$ of measures is uniform. It is sufficient to choose the sequence $\{2^i\}$ for $\{k_i\}$.

Axiom *d).* $\bigcap_{i=1}^{\infty} E_i \notin \mathcal{N}$, *for any sequence* $\{E_i\}_{k=1}^{\infty}$ *of sets such that* $E_i \notin \mathcal{N}_{n_0}$ $(i = 1, 2, 3, \ldots)$.

Theorem 1. *Let* $\{\mathcal{N}_n^t\}$ $(t \in T)$ *be an uniform collection satisfying a), b), c) for every* $t \in T$. *Let* $\{\mathcal{N}_n\}$ *satisfy a), b), c), d). Then* $\{\mathcal{N}_n\} <_t \{\mathcal{N}_n^t\} \Rightarrow \{\mathcal{N}_n\} (<_\tau) \{\mathcal{N}_n^t\}$ *is true.*

Proof. Let $\{k_i\}$ be the sequence of positive integers belonging to the uniform collection $\{\mathcal{N}_n^t\}$. Suppose the theorem does not hold. Then there is n_0 and $\{E_i\}_{i=1}^{\infty}$ such that $E_i \in N_{k_i}^t$ for every t, and $\bigcap_{k=1}^{\infty} \tau^{-k}(E_i) \notin \mathcal{N}_{n_0}$. Putting $F_p = \bigcup_{i=p}^{\infty} E_i$, let $F = \bigcap_{p=1}^{\infty} F_p$. Let N be any natural number. For any $t \in T$ there exists (by c) $r(N, t)$ such that $\bigcup_{i=r(N,t)}^{\infty} E_i \in \mathcal{N}_N^t$. Since $F \subset \bigcup_{i=r(N,t)}^{\infty} E_i \in \mathcal{N}_N^t$ and since N is arbitrary, we have $F \in \mathcal{N}^t$ for every t. Since $\{\mathcal{N}_n\} <_\tau \{\mathcal{N}_n^t\}$, we have

$$\bigcap_{k=1}^{\infty} \tau^{-k}(F) \in \mathcal{N} \tag{2}$$

The inclusion

$$F_p \supset E_p \quad \text{for} \quad p = 1, 2, 3, \ldots$$

gives

$$\tau^{-k}(F_p) \supset \tau^{-k}(E_p), \quad \text{for} \quad p = 1, 2, 3, \ldots$$

Hence

$$\bigcap_{k=1}^{\infty} \tau^{-k}(F_p) \supset \bigcap_{k=1}^{\infty} \tau^{-k}(E_p) \quad \text{for} \quad p = 1, 2, 3, \ldots$$

But $\{\bigcap_{k=1}^{\infty} \tau^{-k}(F_p)\}_{p=1}^{\infty}$ is non-increasing, hence $\bigcap_{p=1}^{\infty} \bigcap_{k=1}^{\infty} \tau^{-k}(F_p) \notin \mathcal{N}$. The last gives

$$\bigcap_{k=1}^{\infty} \tau^{-k}(F) = \bigcap_{k=1}^{\infty} \tau^{-k}\Big(\bigcap_{p=1}^{\infty} F_p\Big) = \bigcap_{k=1}^{\infty} \bigcap_{p=1}^{\infty} \tau^{-k}(F_p) \notin \mathcal{N}$$

This is a contradiction to (2).

Corollary 1. *If* ν *is a finite measure and* $\{\mu_t\}$ $(t \in T)$ *any system of measures such that* $\nu < \{\mu_t\}$ $(t \in T)$, *then* $\nu\ (<)\ \{\mu_t\}$.

The proof of this corollary is given in [4]. If T is an one point set, we have the known theorem ([1] p. 125).

Corollary 2. *If τ is a non-singular transformation satisfying β with respect to $\mathcal{N}$ then, under the assumptions of Theorem* 1, *the following assertions are equivalent:*

$$(i) \quad \{\mathcal{N}_n\} <_\tau \{\mathcal{N}^t_n\} \qquad\qquad (iii) \quad \{\mathcal{N}_n\} < \{\mathcal{N}^t_n\}$$
$$(ii) \quad \{\mathcal{N}_n\} (<_\tau) \{\mathcal{N}^t_n\} \qquad\qquad (iv) \quad \{\mathcal{N}_n\} (<) \{\mathcal{N}^t_n\}$$

Axiom *e*). $\mathcal{N}_n \supset \mathcal{N}_{n+1}$ *for any* n.

In the preceding theorem, the assumption concerning the uniformity of $\{\mathcal{N}^t_n\}$, was used. The following lemma is true:

Lemma 2. *Any countable collection $\{\mathcal{N}^j_n\}$ ($j = 1, 2, 3, \ldots$) satisfying a), b), c), e) is uniform.*

Proof. First of all the condition *e)* implies that $\{k^j_i\}$, for $j = 1, 2, 3, \ldots$, in the condition *c)* may be chosen such that $k^j_i \leqq k^j_{i+1}$. Defining

$$k_i = \max_{j \leqq i} k^j_i \quad (i = 1, 2, 3, \ldots),$$

we have $k_i \geqq k^j_i$ for $i \geqq j$. Hence if $E_i \in \mathcal{N}_{k_i}$, then

$$E_i \in \mathcal{N}^j_{k_i} \quad \text{for any} \quad i \geqq j. \tag{3}$$

Let j be any natural number. Let N be a natural number and $r(N, j)$ such that

$$\bigcup_{i > r(N,j)} F_i \in \mathcal{N}^j_N \quad \text{for any} \quad F_i \in \mathcal{N}^j_{k_i} \tag{4}$$

Denote

$$R(N, j) = \max\,(r(N, j), j).$$

Let $E_i \in \mathcal{N}_{k_i}$ for $i = 1, 2, 3, \ldots$. Put $F_i = \emptyset$ for $i = 1, 2, 3, \ldots\ R(N, j)$, $F_i = E_i$ for $i > R(N, j)$. Evidently $F_i \in \mathcal{N}^j_{k_i}$ (it is sufficient to consider *a), b)* and (3)). Hence, according to (4),

$$\bigcup_{i > R(N,j)} E_i = \bigcup_{i > R(N,j)} F_i \subset \bigcup_{i > r(N,j)} F_i \in \mathcal{N}^j_N.$$

The proof is finished.

There exists a collection satisfying *a), b), c), e)* which is not uniform.

Example 1. Let T denote (in this example) the set of all sequences $\{t_r\}^\infty_{r=1}$ of real numbers such that

$$\lim t_r = 0, \quad t_r \geqq t_{r+1} \ (r = 1, 2, 3, \ldots), \quad 0 < t_r \leqq 1 \tag{5}$$

Let $X = \{1, 2, 3, \ldots\}$ and $\mathcal{S}$ the set of all subsets of X. If $t \in T$, let

$$\mathcal{N}^t_n = \{E : E \in \mathcal{S},\ \sum_{r \in E} t_r \leqq t_n\}$$

for $n = 1, 2, 3, \ldots$

Evidently $\mathcal{N}_n^t$ satisfies *a) b)* and *e)*. It has also the property *c)*. In fact, i f $t \in T$ let k_i^t be such that $\sum_{r=1}^{\infty} t_{k_i^t} < \infty$. Such a sequence $\{k_i^t\}$ exists in view of (5). If N is any natural number, then there is $r(N)$ such that $\sum_{i>r(N)} t_{k_i^t}^t < t_N$. Choosing any sequence $E_i \in \mathcal{N}_{k_i^t}^t$, we have

$$\sum_{\substack{r \in \cup E_i \\ i>r(N)}} t_r \leqq \sum_{i>r(N)} \sum_{r \in E_i} t_r \leqq \sum_{i>r(N)} t_k^t < t_N .$$

Hence $\bigcup_{i>r(N)} E_i \in \mathcal{N}_N^t$. The property *c)* is verified.

The collection $\{\mathcal{N}_n^t\}$ is not uniform. Suppose that a sequence (we may suppose an increasing one) $\{k_i\}$ fulfilling *c)* for any $\{\mathcal{N}_n^t\}$ $(t \in T)$ exists. Choose $t^\circ = \{t_r^\circ\}_{r=1}^{\infty}$ such that

$$\sum_i t_{\kappa i}^\circ = \infty , \quad t^\circ \in T . \tag{6}$$

Choose E_i as a one-point set $\{k_i\}$ for $i = 1, 2, 3, \ldots$. Evidently $E_i \in N_{k_i}^{t^o}$. But for any natural number N

$$\sum_{\substack{r \in \cup E_i \\ i>r(N)}} t_r^\circ = \sum_{i>r(N)} t_{k_i}^\circ = \infty > t_N^\circ .$$

Hence $\{\mathcal{N}_n^{t^o}\}$ does not satisfy *c)*.

If $\{\mathcal{N}_n\}$ is a dominated collection then an analogical theorem to Theorem 1 may be proved without the assumption of the uniformity.

Theorem 2. *Let* $\{\mathcal{N}_n^t\}$ $(t \in T)$ *satisfy a), b), c), e) while,* $\{\mathcal{N}_n^t\} < \{\mathcal{N}_n^*\}$, *where* $\{\mathcal{N}_n^*\}$ *satisfies a), b), c), d) and moreover* $\bigcup_{n=1}^{\infty} \mathcal{N}_n^* = \mathcal{S}$. *Then if* τ *is any measurable transformation,* $\{\mathcal{N}_n\} <_\tau \{\mathcal{N}_n^t\}$ *implies* $\{\mathcal{N}_n\} (<_\tau) \{\mathcal{N}_n^t\}$.

For the proof, the following lemma will be useful.

Lemma 3. *Let* $\{\mathcal{N}_n\}$ *fulfil a), b), c), d) while* $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$. *Then* $\mathcal{S} - \mathcal{N}$ *does not contain an uncountable subset of pairwise disjoint sets.*

Proof. Denote by $\mathcal{E}$ any system of pairwise disjoint sets belonging to $\mathcal{S} - \mathcal{N}$. Then

$$\mathcal{E} = \bigcup_{n=1}^{\infty} (\mathcal{N}_n - \bigcup_{i>n} \mathcal{N}_i) \cap \mathcal{E} . \tag{7}$$

It is sufficient to prove that $(\mathcal{N}_{n_0} - \bigcup_{i>n} \mathcal{N}_i) \cap \mathcal{E}$ is finite for any n_0. Suppose it is not. Then there exists a sequence $\{E_k\}$ of mutually different sets beloning to $(\mathcal{N}_{n_0} - \bigcup_{i>n_0} \mathcal{N}_n) \cap \mathcal{E}$. Put $F_i = \bigcup_{k=i} E_k$. Then $F_i \supset F_{i+1}$ for $i = 1, 2, 3, \ldots$. Since $F_i \supset E_i$, we have $F_i \notin \mathcal{N}_{n_0+1}$. Hence $\bigcap_{i=1}^{\infty} F_i \notin \mathcal{N}$ (by d). The last is a contradiction because $\bigcap_{i=1}^{\infty} F_i = \emptyset$ as it follows from the pairwise disjointness of the sets E_k.

The following well-known result is a corollary of lemma 3.

Corollary. *If μ is a finite measure then any system of pairwise disjoint sets of positive measure is countable.*

Proof. If μ is a probability, it is sufficient to put $\mathcal{N}_n = \left\{E : E \in \mathcal{S},\ \mu(E) \leqq \frac{1}{n}\right\}$. In other cases (with the exception of $\mu \equiv 0$, which is trivial) it is sufficient to put

$$\mu^*(E) = \frac{\mu(E)}{\sup\{\mu(F) : F \in \mathcal{S}\}}.$$

Evidently μ and μ^* have the same sets of positive measure.

Proof of Theorem 2. Since $\{\mathcal{N}_n^t\}$ is dominated by $\{\mathcal{N}_n^*\}$, there exists a countable subsystem $\{\mathcal{N}_n^{t_r}\}$ $(r = 1, 2, 3, \ldots)$ which is equivalent to $\{\mathcal{N}_n^t\}$. Hence $\{\mathcal{N}_n^{t_r}\} \sim \{\mathcal{N}_n^t\}$ ([2], Theorem 3.3). The last gives $\{\mathcal{N}_n\} <_\tau \{\mathcal{N}_n^{t_r}\}$. Hence by Theorem 1, $\{\mathcal{N}_n\} (<_\tau) \{\mathcal{N}_n^{t_r}\}$, so $\{\mathcal{N}_n\} (<_\tau) \{\mathcal{N}_n^t\}$.

The last axiom we shall use is the following one: *f) For any n the following holds: If $E \in \mathcal{N}_n$, $F \in \mathcal{N}$, then $E \cup F \in \mathcal{N}_n$.*

In what follows the systems $\{\mathcal{N}_n\}$ will be supposed to satisfy *a)—f)*.

Lemma 4. *Let $\{\mathcal{N}_n\}$ be given and let $Z \in \mathcal{S}$. Then $\{\mathcal{N}_n^*\}$ defined as $\{\mathcal{N}_n^*\} = \{E : E \in \mathcal{S},\ E \cap Z' \in \mathcal{N}_n\}$ (Z' is a complement of Z) satisfies the axioms a)—f). Moreover if $\bigcup_{n=1}^{\infty} \mathcal{N}_n = \mathcal{S}$, then $\bigcup_{n=1}^{\infty} \mathcal{N}_n^* = \mathcal{S}$.*

Proof. The property *a)* and *b)* for $\mathcal{N}_n^*$ follows from *a)* and *b)* for $\{\mathcal{N}_n\}$. Now let $\{k_i\}$ be the sequence (according to c)) belonging to $\{\mathcal{N}_n\}$. Let N be any natural number. There is $r(N)$ such that $\bigcup_{i>r(N)} E_i \in \mathcal{N}_n$ whenever $E_i \in \mathcal{N}_{k_i}$ $(i = 1, 2, 3, \ldots)$. Let $E_i^* \in \mathcal{N}_{k_i}^*$ i. e. $E_i^* \cap Z' \in \mathcal{N}_{k_i}$. Then $\bigcup_{i>r(N)} E_i^* \cap Z' = Z' \cap \bigcup_{i>r(N)} E_i^* \in \mathcal{N}_N$. Hence $\bigcup_{i>r(N)} E_i^* \in \mathcal{N}_N^*$. The property *c)* for $\mathcal{N}_n^*$ is verified.

Let $E_i \supset E_{i+1}$, $E_i \in \mathcal{S}$ and let $E_i \notin \mathcal{N}_{n_0}^*$ $(i = 1, 2, 3, \ldots)$ then $E_i \cap Z' \in \mathcal{N}_n$

and $d)$ gives $\bigcap_{i=1} E_i \cap Z' \notin \mathscr{N}$. Hence there exists n_1 such that $\bigcap_{i=1} E_i \cap Z' \notin \mathscr{N}_{n_1}$. The last gives $\bigcap_{i=1} E_i \notin \mathscr{N}^*_{n_1}$ thus $\bigcap_{i=1} E_i \notin \mathscr{N}^*$ and $d)$ is proved. The validity of $e)$ follows immediately. Now let n be any natural number, $E \in \mathscr{N}^*_n$, $F \in \mathscr{N}^*$, then $E \cap Z' \in \mathscr{N}_n$, $F \cap Z' \in \mathscr{N}$. We have:

$$(E \cup F) \cap Z' = (E \cap Z') \cup (F \cap Z') \in \mathscr{N}_n.$$

Thus $f)$ is verified.

In view of the last property, it is fufficient to consider instead of the condition $\bigcup_{n=1}^{\infty} \mathscr{N}_n = \mathscr{S}$ the condition $\mathscr{N}_1 = \mathscr{S}$ and to prove $\mathscr{N}^*_1 = \mathscr{S}$. But this is clear since $\mathscr{N}_n \subset \mathscr{N}^*_n \subset \mathscr{S}$ for every n.

Note 9. If $\{\mathscr{N}_n\}$ is a system belonging to a measure μ, then there exists a measure μ^* such that $\{\mathscr{N}^*_n\}$ belongs to μ^*.

Proof. It is sufficient to define $\mu^*(E) = \mu(E \cap Z')$ for $E \in \mathscr{S}$.

Theorem 3. *Let $\{\mathscr{N}^t_n\}$ be a uniform collection such that $\{\mathscr{N}^t_n\} (<\tau) \{\mathscr{N}_n\}$, where $\mathscr{N}_1 = \mathscr{S}$ and τ a non-singular transformation with respect to $\mathscr{N}$ and $\mathscr{N}^t$ for every $t \in T$. Then there exists a system $\{\mathscr{N}^*_n\}$ satisfying $a)$—$f)$ such that $\mathscr{N}^*_1 = \mathscr{S}$ and $\{\mathscr{N}^t_n\} (\sim_\tau) \{\mathscr{N}^*_n\}$.*

Proof. If $\{\mathscr{N}_n\} <_\tau \{\mathscr{N}^t_n\}$ then, according to Theorem 1, $\{\mathscr{N}_n\} (<_\tau) \{\mathscr{N}^t_n\}$, hence $\{\mathscr{N}_n\} (\sim_\tau) \{\mathscr{N}^t_n\}$ (This part of the proof is from the formal point of view not different from the proof of an analogical theorem proved for measure in [4]). Thus, do not let $\{\mathscr{N}_n\} <_\tau \{\mathscr{N}^t_n\}$ be true. In view of the corollary of Theorem 1, $\{\mathscr{N}_n\} < \{\mathscr{N}^t_n\}$ is not true. (We are using a part of the mentioned corollary in which the condition β) is not substantial). The last fact implies the existence of $E \in \mathscr{S}$ such that $E \in \mathscr{N}^t$ for every t and $E \notin \mathscr{N}$. Let $E_r \in \mathscr{S}$ for $r = 1, 2, 3, \ldots$; $E_i \cap E_j = \emptyset$ for $i \neq j$, $E_r \in \mathscr{N}^t$ for every t and let $E_r \notin \mathscr{N}$. Then $\bigcup_{n=1}^{\infty} E_r \in \mathscr{N}^t$ for every t and $\bigcup_{n=1}^{\infty} E_r \notin \mathscr{N}$. Thus the property of the set $E \in \mathscr{S}$ such that $E \in \mathscr{N}^t$ for every t and $E \notin \mathscr{N}$ is invariant under forming countable disjoint unions. Under these assumptions there exists a set $Z \in \mathscr{S}$ such that Z has the mentioned property and any $E \subset Z'$ is such that it has not the above property ([2] Theorem 1.3).

Define $\{\mathscr{N}^*_n\}$ such that for $n = 1, 2, 3, \ldots$. $\{\mathscr{N}^*_n\} = \{E : E \in S, E \cap Z' \in \mathscr{N}_n\}$.

As a consequence of Lemma 4, $\{\mathscr{N}^*_n\}$ has the properties $a)$—$f)$ and $\mathscr{N}^*_1 = \mathscr{S}$ holds. We have $\{\mathscr{N}^*_n\} < \{\mathscr{N}^t_n\}$. In fact, if $E \in \mathscr{N}^t$ for every $t \in T$ then $E \cap Z' \in \in \mathscr{N}^t$ and $E \cap Z' \in \mathscr{N}$ by the propoerty of Z. Hence $E \in \mathscr{N}^*$. The relation $\{\mathscr{N}^*_n\} < \{\mathscr{N}^t_n\}$ gives

$$\{\mathscr{N}^*_n\} (<_\tau) \{\mathscr{N}^t_n\} \tag{8}$$

by the Corollary 2 of Theorem 1. The conditon *β)* again is not important using this part of the corollary. To use the corollary only the non-singularity of τ with respect to $\mathcal{N}^*$ must be verified. It will be done in what follows.

Let $E \in \mathcal{N}^*$. Then $E \cap Z' \in \mathcal{N}$ and $\tau^{-1}(E \cap Z') \in \mathcal{N}$. The last implies

$$Z' \cap [\tau^{-1}(E \cap Z')] \in \mathcal{N}. \tag{9}$$

Since $E \cap Z \subset Z$, we have $E \cap Z \in \mathcal{N}^t$ for every t. The non-singularity of τ with respect to $\mathcal{N}^t$ gives $\tau^{-1}(E \cap Z) \in \mathcal{N}^t$ for every t, hence $[\tau^{-1}(E \cap Z)] \cap$ $\cap Z' \in \mathcal{N}^t$. The last and the property of Z imply

$$[\tau^{-1}(E \cap Z)] \cap Z' \in \mathcal{N}. \tag{10}$$

From (9) and (10) we have

$$\tau^{-1}(E) \cap Z' = \tau^{-1}[(E \cap Z) \cup E \cap Z'] \cap Z' =$$
$$= [\tau^{-1}(E \cap Z)] \cap Z' \cup [\tau^{-1}(E \cap Z'] \cap Z' \in \mathcal{N}.$$

hence $\tau^{-1}(E) \in \mathcal{N}^*$. This proves the non-singularity, hence (8) holds. Now it is sufficient to prove that $\{\mathcal{N}^t_n\} (<_\tau) \{\mathcal{N}^*_n\}$. Let n_0 be any positive integer. There is n' such that if $E \in \mathcal{S}$, $E \in \mathcal{N}_{n'}$ then $E \in \mathcal{N}^t_{n_0}$ for every t. Now let $E \in \mathcal{S}$, $E \in \mathcal{N}_{n'}$. Since $E = (E \cap Z) \cup (E \cap Z')$ and $E \cap Z' \in \mathcal{N}_{n'}$, we have $E \cap Z' \in \mathcal{N}^t_{n_0}$ for every $t \in T$. The fact $E \cap Z \subset Z$ implies $E \cap Z \in \mathcal{N}^t$ for every $t \in T$. Hence

$$E = (E \cap Z) \cup (E \cap Z') \in \mathcal{N}^t_{n_0}$$

by the property *f)*. The proof is finished.

Note 10. The axiom *f)* was used only in the end of the proof for proving that $(E \cap Z') \cup (E \cap Z)$ belongs to $\mathcal{N}^t_{n_0}$. The last fact may be proved also without *f)*. The idea of such a proof is as follows(1): To any number n_0 there exists m_0 (this may be proved without *f*)) such that if $E \in \mathcal{N}^t_{m_0}$, $F \in \mathcal{N}^t$ then $E \cup F \in \mathcal{N}^t_{n_0}$. Now to this m_0 there exists n' such that if $E \in \mathcal{N}_{n'}$ then $E \in \bigcap_{t \in T} \mathcal{N}^t_{m_0}$. $E \in \mathcal{N}^*_{n'}$ implies $E \cap Z' \in \mathcal{N}_{n'}$. This implies $E \cap Z' \in \bigcap_{t \in T} \mathcal{N}^t_{m_0}$. Since $E \cap Z \in \mathcal{N}^t$. we have

$$E = (E \cap Z') \cup (E \cap Z) \in \mathcal{N}^t_{n_0}$$

for every t.

Theorem 3 has the following corollary concerning the probability measures.

(1) This is B. Riečan's idea.

Corollary. *Let* $\{\mu_t\}$ $(t \in T)$ *be a system of probability measures dominated by a probability* μ. *Then there exists a probability measure* p *such that* $\{\mu_t\}(\sim)p$.

Proof. Form the corresponding collection $\{\mathscr{N}_n^t\}$ to the system $\{\mu_t\}$ and the corresponding system $\{\mathscr{N}_n\}$ to μ. By Theorem 3 there exists $\{\mathscr{N}_n^*\}$ such that $\{\mathscr{N}_n^t\}(\sim)\{\mathscr{N}_n^*\}$. It is seen from the construction of $\{\mathscr{N}_n^*\}$ and from the proof of note 9 that $\{\mathscr{N}_n^*\}$ belongs to a finite measure μ^*. Hence $\{\mu_t\}(\sim)\mu^*$. Since $\{\mu_t\}(t \in T)$ are probability measures the measure μ^* may not be identically zero. Thus the measure p defined as $p(E) = \dfrac{\mu^*(E)}{\mu^*(X)}$ is a probability measure and evidently $p(\sim)\mu^*$. The transitivity of the relation $(\sim)$, which is evident, was used here. Later also the transitivity of $(< \tau)$ will be proved.

Theorem 4. *Let* $\{\mathscr{N}_n^t\}$ *and* $\{\mathscr{N}_n\}$, *satisfying a) — f) be given. Let* $\mathscr{N}_1 = \mathscr{S}$ *and* $\{\mathscr{N}_n^t\}(<)\{\mathscr{N}_n\}$. *Then there exists* $\{\mathscr{N}_n^*\}$ *satisfying a) — f) and the condition* $\mathscr{N}_1^* = \mathscr{S}$ *such that* $\{\mathscr{N}_n^t\}(\sim)\{\mathscr{N}_n^*\}$.

Proof. The proof is quite similar to the proof of Theorem 3. The only difference is that Theorem 2 is used in the place where in the proof of Theorem 3 Theorem 1 has been used. Theorem 1 may not be applied because $\{\mathscr{N}_n^t\}$ is not supposed to be uniform. Theorem 2 may be applied because $\{\mathscr{N}_n^t\}$ is dominated. This completes the proof.

If τ-dominated systems of σ-ideals are considered under a suitable transformation τ, then, as we have proved in [2], there exists a countable subsystem τ-equivalent to the given system. It is not possible to prove an analogical theorem for the uniform τ-dominancy. Even for the uniform τ-dominancy of measures such a theorem is not true (see [4]). But it is possible to get from the preceding results abstract formulated analogies of certain necessary and certain sufficient condition for uniform τ-dominancy. The original conditions for the uniform dominancy of measures were proved in [4].

Theorem 5. *A sufficient condition for a collection* $\{\mathscr{N}_n^t\}$ *to be uniformly* τ*-dominated* (τ *is any measurable transformation) by some system is the existence of a countable subsystem* $\{\mathscr{N}_n^{t_i}\}$ $(i = 1, 2, 3, \dots)$ *dominated by a system* $\{\mathscr{N}_n\}$, *such that* $\{\mathscr{N}_n^{t_i}\}(\sim_\tau)\{\mathscr{N}_n^t\}$.

Proof. The only fact we shall need in the proof will be the transitivity of the relation $(<_\tau)$. We shall prove it. Let $\{\mathscr{N}_n^t\}$ $(<_\tau)$ $\{\mathscr{N}_n^v\}$, $\{\mathscr{N}_n^v\}$ $(<_\tau)$ $\{\mathscr{N}_n^w\}$. $(t \in T, v \in V, w \in W)$. To any n_0 there exists n_0' such that $E \in \mathscr{S}$, $E \in \mathscr{N}_{n_0'}^v$ for every $v \in V$ implies $\bigcap_{k=1}^{\infty} \tau^{-k}(E) \in \mathscr{N}_{n_0}^t$ for every $t \in T$. To the number n_0' there exists n_0'' such that if $E \in \mathscr{S}$ and $E \in \mathscr{N}_{n_0''}^w$ for every $w \in W$, then $\bigcap_{k=1}^{\infty} \tau^{-k}(E) \in \in \mathscr{N}_{n_0'}^v$ for every $v \in V$. Let $E \in S$, $E \in \mathscr{N}_{n_0''}^w$ for every $w \in W$. We have

$$\bigcap_{k=1}^{\infty} \tau^{-k}(E) \subset \bigcap_{k=r+1}^{\infty} \tau^{-k}(E) = \tau^{-r}(\bigcap_{k=1}^{\infty} \tau^{-k}(E))$$

for every r.

Therefore

$$\bigcap_{k=1}^{\infty} \tau^{-k}(E) \subset \bigcap_{r=1}^{\infty} \tau^{-r}(\bigcap_{k=1}^{\infty} \tau^{-k}(E)) \in \mathcal{N}^{t}_{n_0}$$

for every $t \in T$, because $\bigcap_{k=1}^{\infty} \tau^{-k}(E) \in \mathcal{N}^{v}_{n_0'}$ for every $v \in V$. The transitivity is proved. The proof of the transitivity of $<_\tau$ is quite analogical. From this the proof of Theorem 5 immediately follows.

Theorem 6. *Let $\{\mathcal{N}^t_n\}$ $(t \in T)$ be a uniform collection such that $\mathcal{N}^t_n$ $(\sim_\tau)$ $\{\mathcal{N}_n\}$ and $\mathcal{N}_1 = \mathcal{S}$. Let τ be a measurable transformation having the properties: α) for every $\mathcal{N}^t$ $(t \in T)$, β) with respect to $\mathcal{N}$ and γ) for every $\mathcal{N}^t$ $(t \in T)$. Then there exists a countable subcollection $\{\mathcal{N}^{t_i}_n\}$ of the collection $\{\mathcal{N}^t_n\}$ such that $\{\mathcal{N}^t_n\}$ $(\sim_\tau)$ $\{\mathcal{N}^{t_i}_n\}$ and $\{\mathcal{N}^{t_i}_n\}$ $(<_\tau)$ $\{\mathcal{N}_n\}$.*

Proof. By the assumption

$$\{\mathcal{N}^t_n\} \ (\sim_\tau) \ \{\mathcal{N}_n\}. \tag{11}$$

From this evidently

$$\{\mathcal{N}^t_n\} <_\tau \{\mathcal{N}_n\} \tag{12}$$

and by Theorem 3.3 from [2], there is a countable subcollection $\{\mathcal{N}^{t_i}_n\}$ $(i = 1, 2, 3, \ldots)$ of the collection $\{\mathcal{N}^t_n\}$ such that

$$\{\mathcal{N}^{t_i}_n\} \sim_\tau \{\mathcal{N}^t_n\}. \tag{13}$$

Since $\{\mathcal{N}^{t_i}_n\} \subset \{\mathcal{N}^t_n\}$, we have

$$\{\mathcal{N}^{t_i}_n\} \ (<_\tau) \ \{\mathcal{N}_n\} \tag{14}$$

(11) and (13) imply $\{\mathcal{N}_n\} < \{\mathcal{N}^{t_i}_n\}$ and Theorem 1 gives $\{\mathcal{N}_n\}$ $(<_\tau)$ $\{\mathcal{N}^{t_i}_n\}$. The last and (14) give $\{\mathcal{N}^{t_i}_n\}$ $(\sim_\tau)$ $\{\mathcal{N}_n\}$. From this and from (11) we get $\{\mathcal{N}^{t_i}_n\}$ $(\sim_\tau)$ $\{\mathcal{N}^t_n\}$.

Theorem 7. *A necessary and sufficient condition for a collection $\{\mathcal{N}^t_n\}$ (not necessarily uniform) to be uniformly dominated by a system $\{\mathcal{N}_n\}$ satisfying $\mathcal{N}_1 = \mathcal{S}$ is the existence of a countable subcollection $\{\mathcal{N}^{t_i}_n\}$ of $\{\mathcal{N}^t_n\}$ such that $\{\mathcal{N}^t_n\}$ $(\sim)$ $\{\mathcal{N}^{t_i}_n\}$ and such that $\{\mathcal{N}^{t_i}_n\}$ is uniformly dominated by $\{\mathcal{N}^*_n\}$ satisfying $\mathcal{N}^*_1 = \mathcal{S}$,*

Proof. The sufficiency follows from Theorem 5. The proof of the neccessity will be acomplished in what follows. Let $\{\mathcal{N}_n^t\}\ (<)\ \{\mathcal{N}_n\}$, $\mathcal{N}_1 = \mathcal{S}$. In view of Theorem 4, there exists $\{\mathcal{N}_n^*\}$ such that

$$(15) \qquad \{\mathcal{N}_n^t\} \sim \{\mathcal{N}_n^*\},\ \mathcal{N}_1^* = \mathcal{S}.$$

Using the result 3.3 from [2] in the same way as in the proof of Theorem 6, we have a subcollection $\{\mathcal{N}_n^{t_i}\} \subset \{\mathcal{N}_n^t\}$ such that

$$(16) \qquad \{\mathcal{N}_n^{t_i}\}\ (\sim)\ \{\mathcal{N}_n^t\}.$$

Evidently

$$(17) \qquad \{\mathcal{N}_n^{t_i}\}\ (<)\ \{\mathcal{N}_n^*\}$$

(15) and (16) give $\{\mathcal{N}_n^*\} < \{\mathcal{N}_n^{t_i}\}$ and using Theorem 2, we get

$$(18) \qquad \{\mathcal{N}_n^*\}\ (<)\ \{\mathcal{N}_n^{t_i}\}.$$

By (17) and (18) $\{\mathcal{N}_n^{t_i}\}\ (\sim)\ \{\mathcal{N}_n^*\}$. Hence, in view of (15), $\{\mathcal{N}_n^{t_i}\}\ (\sim)\ \{\mathcal{N}_n^t.\}$ The Theorem is proved.

Corollary. ([4] Theorem 2). *A necessary and sufficient condition for a system $\{\mu_t\}$ $(t \in T)$ of measures to be uniformly dominated by a probability measure p is the existence of a countable subsystem $\{\mu_{t_i}\}$ $(i = 1, 2, 3, \ldots)$ uniformly dominated by a finite measure μ and uniformly equivalent to $\{\mu_t\}$.*

Note. For the proof it is sufficient to take a probability measure which dominates $\{\mu_t\}$. Such a measure may be constructed by μ which dominates $\{\mu_t\}$, and then the Theorem may be applied.

2

In this part, the connections between the system of axioms used here and that used in [3], will be studied.

In [3] the following system of axioms was used.

(i) $\emptyset \in \mathcal{N}_n$ $(n = 1, 2, 3, \ldots)$.

(ii) For every natural n tere exists a sequence $\{k_i\}$ of natural numbers such that $\bigcup_{i-1}^{\infty} E_i \in \mathcal{N}_n$ whenever $E_i \in \mathcal{N}_{k_i}$.

(iii) If $\{E_i\}$ is a sequence of sets in $\mathcal{S}, E_{i+1} \subset E_i$, $(i = 1, 2, 3, \ldots)$, $\bigcap_{i=1}^{\infty} E_i = \emptyset$, then for every n there exists m such that $E_m \in \mathcal{N}_n$.

(iv) If $E \in \mathcal{N}_n$, $F \subset E$, $F \in \mathcal{S}$, then $F \in \mathcal{N}_n$.

(v) If $E \in \bigcap_{n=1}^{\infty} \mathcal{N}_n$, $F \subset E$, then $F \in \bigcap_{n=1}^{\infty} \mathcal{N}_n$.

The axiom (v) was used only in the case of questions concerning a complete measure. Therefore it is of a special type and can not be deduced from $a) - f)$. But if the system (i) — (iv) is considered and on the other hand the system $a) - f)$, then any system $\{\mathcal{N}_n\}$ belonging to any finite measure μ evidently satisfies these systems of axioms.

Now the connections between the mentioned systems will be discussed. The questions whether the axioms are independent will not be studied for any of the systems.

Lemma 5. *Let* $\{\mathcal{N}_n\}$ *satisfy* $a) - f)$. *Then* (i) — (iv) *are satisfied. There exists a system satisfying* (i) — (iv), *and failing to satisfy* $a) - f)$.

Proof. The property (i) follows from $a)$ and $b)$, (iii) follows from $d)$; (iv) coincides with $b)$. There remains (ii) to be proved. Let n be any natural number. Let $\{k'_i\}$ be a sequence of natural numbers the existence of which follows from $c)$. Put $N = n$ in $c)$ and take the corresponding $r(n)$. Let $k_1 = k'_{r(n)+1}$, $k_2 = k'_{r(n)+2}, \ldots$ If $E_i \in \mathcal{N}_{k_i}$, then $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i>r(n)} E_i \in \mathcal{N}_n$. Hence $\{k_i\}$ has the property required by (ii).

Let $X = \{a, b\}$ be a two-point set. Put

$$\mathcal{S} = \{\{a\}, \{b\}, \{a, b\}, \emptyset\}; \quad \mathcal{N}_1 = \mathcal{S}; \ \mathcal{N}_2 = \{\{a\}, \{b\}, \emptyset\};$$
$$\{\emptyset, \{a\}\} = \mathcal{N}_3 = \mathcal{N}_4 = \ldots$$

$\{\mathcal{N}_n\}$ satisfies (i) — (iv) but fails to satisfy $f)$.

Adjoining the axioms $e), f)$ to (i) — (iv) the systems become equivalent.

Theorem 8. *The systems* (i), (ii), (iii), (iv), $e), f)$ *and* $a), b), c), d), e), f)$ *are equivalent. The axioms* $e), f)$ *adjoined to* (i) — (iv) *are independent on* (i) — (iv).

Proof. By Lemma 5, if $\{\mathcal{N}_n\}$ satisfies $a) - f)$, then it satisfies (i) — (iv), $e), f)$. Now let $\{\mathcal{N}_n\}$ satisfy (i) — (iv), $e), f)$. We shall show that $\{\mathcal{N}_n\}$ satisfies $c)$ and $d)$. The fact that it satisfies $a), b), e), f)$ is trivial.

Let us prove $d)$ first. Let $\{E_i\}$ be a non-increasing sequence of sets such that $E_i \in \mathcal{S}$, $E_i \notin \mathcal{N}_{n_0}$ $(i = 1, 2, 3, \ldots)$. Let $F = \bigcap_{i=1}^{\infty} E_i$. Suppose $F \in \mathcal{N}$. Then $F \cap F_i \in \mathcal{N}$ for every i. Hence (by $f)$) $E_i - F \notin \mathcal{N}_{n_0}$. Using (iii), we get $\bigcap_{i=1}^{\infty} (E_i - F) \neq \emptyset$. This is a contradiction.

Now we shall prove $c)$. First of all, the property $e)$ implies that $\{k_i\}$ in (ii) may be chosen as an increasing sequence. Let $N_1 < N_2 < N_3 \ldots$ be an increasing sequence of natural numbers. For $r = 1, 2, 3, \ldots$ let $\{k_i^r\}$ be the corresponding increasing sequences. Put $k_i = \max_{j \leq i} k_i^j$. Now, let $E_i \in \mathcal{N}_{k_i}$ $(i = 1, 2, 3, \ldots)$ and let r be a natural number. For $i \geqslant r$, $\mathcal{N}_{k_i} \subset \mathcal{N}^r_{k_i}$. Hence $E_i \in \mathcal{N}^r_{k_i}$ if $i \geqslant r$. If we put empty sets instead of the first $r - 1$ terms in

the sequence $\{E_i\}$, then for a new sequence $\{F_i\}$ which will be obtained, $(F_i = \emptyset,\ i \leqq r - 1;\ F_i = E_i$ if $i \geqslant r)$ $F_i \in \mathcal{N}_{k_i}$ holds.
Hence

$$\bigcup_{i \geqq r}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i \in \mathcal{N}_{N_r}.$$

Thus, if N is any natural number, it is sufficient to choose r such that $N_r > N$ and we have $\bigcup_{i \geqq r} E_i \in \mathcal{N}_N$.

Now it remains to be proved that one cannot deduce $e), f)$ from (i) — (iv). As to $f)$, it follows from the example in the proof of Lemma 5. For $e)$ it is sufficient to transpose $\mathcal{N}_1$ and $\mathcal{N}_2$ in the mentioned example.

Let us remark at the end that a system satisfying (i) — (iv), $e), f)$ and not belonging to any additive (and what is more to any superadditive function), exists. More precisely There exists a system $\{\mathcal{N}_n\}$ satisfying the mentioned axioms such that there is no superadditive function λ for which $\mathcal{N}_n = \left\{E : E \in \mathcal{S}, \lambda(E) < \frac{1}{n}\right\}$ It is sufficient to put

$$X = \{a, b\};\ \mathcal{S} = \{\{a\}, \{b\}, \{a, b\}, \emptyset\};\ \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{S};\ \{\emptyset\} = \mathcal{N}_3 = \mathcal{N}_4 = \ldots$$

The axioms are evidently satisfied. A function λ with the above mentioned property does not exist because if there is such a λ, then $\lambda(\{a\}) \geqslant \frac{1}{3}$ $\lambda(\{b\}) \geqslant \frac{1}{3}$, hence $\lambda(\{a, b\} \geqslant \lambda(\{a\}) + \lambda(\{b\}) \geqslant \frac{2}{3}$ which contradicts the fact, $\{a, b\} \in \mathcal{N}_2$.

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