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MONOTONE AND OSCILLATORY SOLUTIONS OF A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS

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The aim of this paper is to investigate some properties of the solutions of the equation

$$(r(x)y'(x))' + p(x, y(x), y'(x)) = 0, \tag{r}$$

where $r(x)$, $p(x, u, v)$ are functions satisfying the following conditions

1. $r(x) \in C^1 < a, \infty)$, $r(x) > 0$ for every $x \in < a, \infty)$, where $\alpha(x) = \int\limits_a^x 1/r(t)\mathrm{d}t$.

2. $p(x, u, v) \in C^\circ$ in some 3-dimensional region, which will be specified in the following theorems. If nothing else is said it will be a region

$$D\colon\ a \leqslant x < \infty,\ -\infty < u < \infty,\ -\infty < v < \infty.$$

In some theorems these assumptions will be completed with condition

3. For every point $(x, u, v) \in D$, $u \neq 0$ $p(x, u, v)u > 0$.

Further assumptions will be done in single theorems.

By the solution of the equation (r) we understand only a solution defined in some interval $\langle x_0, \infty)$ $(x_0 \geqslant a)$. A solution $y(x)$ will be called oscillatory if it has at least one zero in the interval (x, ∞) for an arbitrary x. In the opposite case this solution will be called nonoscillatory.

This paper is divided into two parts. The first part deals with the existence of nonoscillatory bounded solutions of the equation (r) and solutions of the form $y(x) \sim c\alpha(x)$. It is proved further that (r) under some additional conditions has no other nonoscillatory solutions besides the solutions of the given form. The second part deals with the oscillatory solutions of (r). There are given some sufficient conditions, in order that all solutions of (r) may be oscillatory. In some cases these conditions are necessary and sufficient. There are given further some theorems concerning the increase or decrease of the „amplitudes" of oscillatory solutions and sufficient conditions in order that the equation (r) may not have oscillatory solutions, besides a trivial one.

The special forms of (r) were studied by several authors, e. g. in the papers [1], [7], [3], [8], [2], [9] and many others. This paper generalizes some of the results of these authors.

I.

Theorem 1. *Let condition 2. be satisfied in the region*

$$D_1:\ a \leqslant x < \infty,\ u_0 \leqslant u \leqslant u_1,\ 0 \leqslant v \leqslant v_1.$$

Let the function $p(x, u, v)$ be non-negative and non-decreasing in u, v on D_1, for every fixed x.

If for some constants c_0 and $c_1 (u_0 < c_0 \leqslant u_1, 0 < c_1 \leqslant v_1)$ $\int^{\infty} \alpha(x)p(x, c_0, c_1/r(x))\mathrm{d}x < \infty$ holds, then for every m, $u_0 \leqslant m < c_0$ there exists $b_0(m)$ such that for all $b \geqslant b_0(m)$ there exists a solution of (r) *defined at least in the interval $\langle b, \infty)$, passing through the point (b, m) and monotonely increasing to a constant $c \leqslant c_0$.*

Conversely, if (r) *has such a solution, then for arbitrary numbers c_2, c_3 such that $u_0 \leqslant c_2 < c$, $0 \leqslant c_3 \leqslant r(\infty)y'(\infty)$, we have $\int^{\infty} a(x)p(x, c_2, c_3/r(x))\mathrm{d}x < \infty$.*

Proof. 1. Let c_0 and c_1 be such constants and let m be an arbitrary number satisfying $u_0 \leqslant m < c_0$. Then there exists $b_0(m) \geqslant a$ such that for every $b \geqslant b_0(m)$ we have

$$\int_b^{\infty} \alpha(x)p(x, c_0, c_1/r(x))\mathrm{d}x \leqslant c_0 - m, \tag{1}$$

$$\int_b^{\infty} p(x, c_0, c_1/r(x))\mathrm{d}x \leqslant c_1.$$

Consider the equation

$$y(x) = m + \int_b^x \{\alpha(t) - \alpha(b)\}p(t, y(t), y'(t))\mathrm{d}t + \tag{2}$$
$$+ \{\alpha(x) - \alpha(b)\}\int_x^{\infty} p(t, y(t), y'(t))\mathrm{d}t.$$

We prove that the equation (2) has a solution $y(x)$ passing through the point (b, m) and monotonely increasing to some constant $c \leqslant c_0$. This solution is also a solution of the equation (r). The existence of a solution of (2) will be proved by the method of successive approximations. If we put $y_1(x)$ m adn for $n = 1, 2, 3, \ldots$

$$y_{n+1}(x) = m + \int_b^x \{\alpha(t) - \alpha(b)\}p(t, y_n(t), y_n'(t))\mathrm{d}t + \tag{3}$$

$$+ \{\alpha(x) - \alpha(b)\}\int_x^\infty p(t, y_n(t), y_n'(t))\mathrm{d}t,$$

then for $x \geqslant b$ and for every n the following inequalities hold

$$m \leqslant y_n(x) \leqslant c_0, \quad 0 \leqslant y_n'(x) \leqslant c_1/r(x), \tag{4}$$

which may be proved by induction, using (1). Similarly, it may be proved by induction that for every $x \geqslant b$, the sequences $\{y_n(x)\}$, $\{y_n'(x)\}$ are non-decreasing. Thus there exists $y(x)$ such that for every $x \in \langle b, \infty)$ we have $\lim_{n\leftarrow\infty} y_n(x) = y(x)$. Evidently, the functions $\{y_n(x)\}$ and $\{y_n'(x)\}$ are uniformly bounded and equicontinuous on every finite interval. Thus, on this intervals $\lim_{n\to\infty} y_n(x) = y(x)$, $\lim_{n\to\infty} y_n'(x) = y'(x)$ uniformly, where $y(x)$ and $y'(x)$ satisfy (4). Using these considerations and the Lebesgue theorem we get by (3) that $y(x)$ is a solution of equation (2). This solution exists at least in the interval $\langle b, \infty)$ and has the required properties.

2. Let $y(x)$ be a solution of (r) considered in the first part. Then there exists a number $b \geqslant a$ such that for $x \geqslant b$, $c \geqslant y(x) \geqslant c_2$ and $r(x)y'(x) \geqslant$ $\geqslant r(\infty)\, y'(\infty) \geqslant 0$. Now, from (r) and the last inequalities we get

$$c \geqslant y(x) \geqslant y(b) + \int_b^x \{\alpha(t) - \alpha(b)\}p(t, c_2, c_3/r(t))\mathrm{d}t,$$

for all $x \geqslant b$. From this, $\int^\infty \alpha(x)p(x, c_2, c_3/r(x))\mathrm{d}x < \infty$, which proves the theorem.

In the following theorem we omit the assumption of monotony of the function $p(x, u, v)$. Here and in some of the next theorems we shall consider two functions $f_1(x, u, v)$, $f_2(x, u, v)$ and it will be supposed they are continuous and non-decreasing in u and v, for every fixed $x \geqslant a$ on some region.

Theorem 2. *Let there exist the functions $f_1(x, u, v)$ and $f_2(x, u, v)$ such that for every point of D_1*

$$0 \leqslant f_1(x, u, v) \leqslant p(x, u, v) \leqslant f_2(x, u, v). \tag{5}$$

Denote

$$(r(x)z'(x))' + f_2(x, z(x), z'(x)) = 0 \tag{r_2}$$

$$(r(x)w'(x))' + f_1(x, w(x), w'(x)) = 0. \tag{r_1}$$

If there exist constants c_0, c_1, ($u_0 < c_0 \leqslant u_1$, $0 < c_1 \leqslant v_1$) such that

$$\int^\infty \alpha(x)f_2(x, c_0, c_1/r(x))\mathrm{d}x < \infty, \tag{6}$$

then for every m, $u_0 \leqslant m < c_0$, there exists $b_0(m)$ such that for all $b \geqslant b_0$ through *the point* (b, m) *a solution of* (r) *passes defined at least in the interval* $\langle b, \infty)$. *By Theorem* 1 *this solution lies between the solutions* $z(x)$ *and* $w(x)$ *of* (r_2) *and* (r_1), *passing through this point. Conversely, if* (r) *has such a solution, then* $\int^{\infty} \alpha(x) f_1 (x, c_2, c_3/r(x))dx < \infty$, *where* c_2 *and* c_3 *are such as in Theorem* 1.

This theorem will be proved by the following particular case of Tychonov's theorem.

Lemma. *Let* X *be a linear metric, locally convex, complete space (i. e. Fréchet space). Let* M *be a convex, closed subset of* X. *If* T *is a continuous operator of* M *into itself, such that the closure of* TM *is a compact subset of* M, *then there exists at least one fixed point of* T, (see [4]).

Let X be a space of continuuous, bounded functions $f(x)$ with continuous derivatives, such that $r(x)f'(x)$ are bounded on the interval $I = \langle b, \infty)$. Let a sequence $\{x_n\} \to \infty$ for $n \to \infty$ be such that $b = x_0 < x_1 < x_2 \dots$. Let us denote $K_n(f) = \max_{x \in <b, x_n>} |f(x)| + \max_{x \in <b, x_n>} r(x)|f'(x)|$. Then the system of seminorms $K_n(f)$ defines a topology of X, under which X is locally convex. The space X is metrizable as well and the convergence on it is the uniform convergence of the functions and their first derivatives on every compact subinterval of I. Thus X is a Frèchet space, (see [11]).

Proof of Theorem 2. From the construction of solutions $w(x)$ and $z(x)$ passing through the point (b, m), by Theorem 1 and (5) it follows that $w(x) \leqslant z(x)$, $w'(x) \leqslant z'(x)$. Define a set $M \subset X$ and an operator T in the following way

$$M = \{f(x) \in X : w(x) \leqslant f(x) \leqslant z(x),\ w'(x) \leqslant f'(x) \leqslant z'(x)\}, \tag{7}$$

$$Tf(x) = m + \int_a^x \{\alpha(t) - \alpha(b)\} p(t, f(t), f'(t))dt + \\ + \{\alpha(x) - \alpha(b)\} \int_x^{\infty} p(t, f(t), f'(t))dt.$$

The set M and the operator T have the following properties:

1. M is convex and closed, which can be easily proved.
2. Operator T maps M into itself. Let $f(x) \in M$, then by (5) and (7)

$$Tf(x) \leqslant m + \int_b^x \{\alpha(t) - \alpha(b)\} f_2(t, z(t), z'(t))dt + \\ + \{\alpha(x) - \alpha(b)\} \int_x^{\infty} f_2(t, z(t), z'(t))dt = z(x).$$

The inequalities $w(x) \leqslant Tf(x)$ and $w'(x) \leqslant (Tf(x))' \leqslant z'(x)$ are proved similarly.

3. The continuity of T can be easily proved by (6) and the Lebesgue theorem.

4. Let us denote $TM = \{f_\beta(x)\}$. Then this set of functions and the set $\{r(x)f'_\beta(x)\}$ as well are uniformly bounded on the interval I, which follows from the definition of M and the property 2. It will be proved that these sets are also equicontinuous on I. Let $\varepsilon > 0$ be an arbitrary number, then there exists $c > b$, such that $\int\limits^{\infty} f_2(x, c_0, c_1/r(x))\mathrm{d}x < \varepsilon/2$. Let us divide the interval $\langle b, \infty)$ into two subintervals $\langle b, c\rangle$ and $\langle c, \infty\rangle$. On the interval $\langle b, c\rangle$ the set $\{r(x)f'_\beta(x)\}$ is equicontinuous, because $|r(x_2)f'_\beta(x_2) - r(x_1)f'_\beta(x_1)| \leqslant |\int\limits_{x_1}^{x_2} f_2(x, c_0, c_1/r(x))\mathrm{d}x|$. If the numbers $x_1, x_2 \in \langle c, \infty)$, then $|r(x_2)f'_\beta(x_2) - r(x_1)f'_\beta(x_1)| \leqslant \int\limits_c^{\infty} f_2(x, x_0, c_1/r(x))\mathrm{d}x < \varepsilon$. In the case of $x_1 \in \langle b, c\rangle$, $x_2 \in \langle c, \infty)$, then $|r(x_2)f'_\beta(x_2) - r(x_1)f'_\beta(x_1)| \leqslant |r(x_2)f'_\beta(x_2) - r(c)f'_\beta(c)| + |r(c)f'_\beta(c) - r(x_1)f'_\beta(x_1)| < {}^{\varepsilon}/_2 + {}^{\varepsilon}/_2 = \varepsilon$. Thus the considered set is equicontinuous on I. If we use (5) and (6), we can prove by a similar consideration that the set $\{f_\beta(x)\}$ is also equicontinuous on the interval I. Consequently, Arzela's theorem shows that TM is a compact subset of M.

All assumptions of the Lemma are satisfied, thus T has a fixed point on M, i. e. the equation (r) has a solution $y(x)$, passing through the point (b, m) and such that $w(x) \leqslant y(x) \leqslant z(x)$ for all $x \geqslant b$. This proves the first part of our theorem. The proof of the second part is similar to that of Theorem 1.

Remark 1. Let in the region $D_2 = \{a \leqslant x < \infty, u_0 \leqslant u \leqslant u_1, v_0 \leqslant v \leqslant 0\}$ $p(x, u, v) \leqslant 0$ and $\int\limits^{\infty} \alpha(x)\,|p(x, c_0, c_1/r(x))|\,\mathrm{d}x < \infty$ (c_0, c_1 are suitable constants), then the conclusions of Theorem 1 remain valid, up to the fact, that the solution is monotonely decreasing. The conclusions of Theorem 2 remain also valid if instead of (5) we demand

$$f_2(x, u, v) \leqslant p(x, u, v) \leqslant f_1(x, u, v) \leqslant 0 \tag{8}$$

for every point of D_2 and $\int\limits^{\infty} \alpha(x)\,|f_2(x, c_0, c_1/r(x))|\,\mathrm{d}x < \infty$. The proofs are evident from those of Theorems 1 and 2.

The next two theorems deal with such solutions of (r), for which the $\lim\limits_{x\to\infty} y(x)/\alpha(x) = c$, i. e. $y(x) \sim c\alpha(x)$, $c \neq 0$.

Theorem 3. *Let the function $p(x, u, v)$ be non-negative, non-decreasing in u and v on $D_3 = \{a \leqslant x < \infty, u_0 \leqslant u < \infty, 0 \leqslant v \leqslant v_1\}$ for every fixed x and let $\lim\limits_{y\to\infty} \alpha(x) = \infty$. If there exist positive numbers c_0 and c_1 such that $v_1 \geqslant c_1 \geqslant c_0 > u_0$ and*

$$\int\limits^{\infty} p(x, c_0\alpha(x), c_1/r(x))\mathrm{d}x < \infty, \tag{9}$$

then (r) *has a solution $y(x) \sim c\alpha(x)$, where $u_0 < c \leqslant c_0$.*

Conversely, if (r) *has such a solution, then for arbitrary numbers* c_2, c_3 *such that* $u_0 \leqslant c_2 < c$, $0 \leqslant c_3 \leqslant r(\infty)y'(\infty)$, *we have* $\int^{\infty} p(x, c_2\alpha(x), c_3/r(x))\mathrm{d}x < \infty$.

Proof. 1. Let us consider the equation

$$(10) \qquad W(x) = m + (1/\alpha(x))\int_b^x \alpha(t)p(t, \alpha(t)W(t), (\alpha(t)W(t))')\mathrm{d}t + \\ + \int_x^{\infty} p(t, \alpha(t)W(t), (\alpha(t)W(t))')\mathrm{d}t,$$

where $u_0 \leqslant m < c_0$. It will be proved that this equation has a solution $Y(x) \sim c$. Let $\{W_n(x)\}$ be successive approximations for (10), where $W_1(x) = m$. If two numbers c_0 and c_1 are given, then by (9) there exists a number b such that $\int_b^{\infty} p(x, c_0\alpha(x), c_1/r(x))\mathrm{d}x \leqslant c_0 - m$. Let us suppose that the number b in (10) is chosen in this manner. Then for all $x \geqslant b$ and an arbitrary positive integer we have

$$(11) \qquad m \leqslant W_n(x) \leqslant c_0, \ 0 \leqslant (\alpha(x)W_n))' \leqslant c_1/r(x),$$

where these sequences are non-decreasing. The proof of these assertions can be easily made by induction. From (11) and (9) it follows that the sequence $\{(\alpha(x)W_n(x))'\}$ is uniformly bounded and equicontinuous on every finite interval. Thus there exists a function $Z(x)$ such that for all $x \geqslant b$, $\lim_{n\to\infty} (\alpha(x)W_n(x))' = Z(x)$ uniformly on every finite interval. For the sequence $\{W_n(x)\}$ we have similarly $\lim_{n\to\infty} W_n(x) = Y(x) \leqslant c_0$. By the well-known classical theorem it follows that for all $x \geqslant b$, $Z(x) = (\alpha(x)Y(x))'$. From the preceding considerations and Lebesgue's dominated convergence theorem we get that $Y(x)$ is a solution of (10). Let us denote $y(x) = \alpha(x)Y(x)$, then by (10) it follows that $y(x)$ is a solution of (r), where $y(x) \sim c\alpha(x)$.

2. Let $y(x)$ be such a solution. Then for every number c_2 ($u_0 \leqslant c_2 < c$) there exists $b \geqslant a$ such that in the interval $\langle b, \infty)$ $y(x) \geqslant c_2\alpha(x)$ and $r(x)y'(x) \geqslant$ $\geqslant r)\infty)y'(\infty) \geqslant 0$. From (r) and the preceding inequalities we have

$$r(b)y'(b) \geqslant \int_b^x p(t, y(t), y'(t))\mathrm{d}t \geqslant \int_b^x p(t, c_2\alpha(t), c_3/r(t))\mathrm{d}t.$$

This implies that $\int^{\infty} p(t, c_2\alpha(t), c_3/r(t))\mathrm{d}t < \infty$ and the proof of the theorem is complete.

Similarly as in Theorem 2, we omit in the following theorem the assumption of monotony of the function $p(x, u, v)$.

Theorem 4. *Let $f_1(x, u, v)$ and $f_2(x, u, v)$ be such functions that for every point of D_3 inequality* (5) *is satisfied. Let further $\lim_{x\to\infty} \alpha(x) = \infty$ and let there exist positive numbers c_0, c_1 ($v_1 \geqslant c_1 \geqslant c_0 > u_0$) such that $\int^{\infty} f_2(x, c_0\alpha(x), c_1/ /r(x)\mathrm{d}x < \infty$. Then* (r) *has a solution $y(x) \sim c\alpha(x)$. This solution lies for all sufficiently large x between the solutions $w(x)$ and $z(x)$ of* (r_1) *and* (r_2) *that have the same asymptotic behaviour and their existence is guaranted by Theorem* 3. *Conversely, if* (r) *has such a solution, then $\int^{\infty} f_1(x, c_2\alpha(x), c_3/r(x))\mathrm{d}x < \infty$, where c_2 and c_3 are the same as in Theorem* 3.

This theorem can be proved in a similar way as Theorem 2. If we suppose e. g. X to be a space of all continuously differentiable bounded functions such that $r(x)\alpha(x)f'(x)$ are bounded on $I = \langle b, \infty)$, then we define T by (10) and M is defined as follows

$$M = \{f(x) \in X : w(x) \leqslant \alpha(x)f(x) \leqslant z(x), w'(x) \leqslant (\alpha(x)f(x))' \leqslant z(x)\}.$$

Remark 2. From the proof of Theorem 3 it is evident that the conclusions of this Theorem remain valid also in the case if $p(x, u, v) \leqslant 0$ in the region $D_4 = \{a \leqslant x < \infty, -\infty < u \leqslant u_0, v_0 \leqslant v \leqslant 0\}$ and $\int^{\infty} |p(x, c_0\alpha(x), c_1/r(x))|\mathrm{d}x < < \infty$ (c_0 and c_1 are suitable constants). Similarly if in Theorem 4 instead of (5) inequality (8) holds for every point of D_4 and all the other of the assumptions are satisfied, then the conclusions of the last theorem remain valid.

From the preceding theorems it follows that equation (r) can have bounded as well unbounded solutions. However in the case of boundedness of $\alpha(x)$ we have

Theorem 5. *Let the function $p(x, u, v) \geqslant 0$ in the region $D_5 = \{a \leqslant x < \infty, 0 \leqslant u < \infty, 0 \leqslant v < \infty\}$ and let $\lim_{x\to\infty} \alpha(x) < \infty$. Then every solution of* (r) *in D_5 is bounded.*

Let there exist two functions $f_1(x, u, v)$ and $f_2(x, u, v)$ satisfying (5) *in D_5. Let there exists a number $c_1 > 0$ such that for every $0 < c_0 \leqslant c_1$ and all sufficiently large u and x*

$$f_2(x, c_0u, c_1/r(x))/u \leqslant Kf_1(x, c_0, 0), \tag{12}$$

where K is a constant. If all solutions of (r) *are bounded, then $\alpha(x)$ is also bounded.*

Proof. 1. This part is evident, because if $y(x)$ is a solution of (r), then there exists b such that for $x \geqslant b$ from (r) we have $0 \leqslant y(x) \leqslant \int_b^x (r(b)y'(b)/ / r(t))\mathrm{d}t + y(b)$.

2. Let $\alpha(x)$ be unbounded and all solutions of (r) be bounded. Then by Theorem 2, $\int^{\infty} \alpha(x) f_1(x, c_0, 0) \mathrm{d}x < \infty$. But from (12) we have $\int^{\infty} \alpha(x) f_2(x, c_0\alpha(x), c_1/r(x)) \mathrm{d}x < \infty$ and this, by Theorem 4, implies the existence of a solution $y(x) \sim c\alpha(x)$ of (r), which is a contradiction.

In the next two theorems it will be proved that (r), under some assumptions, has no other solutions, besides the ones given in Theorems 1 and 3. In all of the following theorems, besides Theorems 16, 17 and 18, it will be supposed that $\lim\limits_{\alpha\to\infty} \alpha(x) = \infty$.

Theorem 6. *Let for all points of D_5 $p(x, u, v) > 0$, continuous and non-decreasing in v. Let there exist c_0 such that for $u \in \langle c_0, \infty)$ $p(x, u, v)/u$ be non-increasing in u for every x and v. Besides, if for every number $c_1 > 0$ we have*

$$\int^{\infty} \alpha(x) p(x, c_0, c_1/r(x)) \mathrm{d}x < \infty, \tag{13}$$

then all solutions of (r) *in D_5 are either bounded or of then form $y(x) \sim c\alpha(x)$.*

Proof. Let $y(x)$ be an unbounded solution of (r), then there exist numbers $c_1, b_1 \geqslant a$ such that for all $x \geqslant b_1$, $y(x) > c_0$ and $y'(x) \leqslant c_1/r(x)$. Integrating (r) and using the assumption $p(x, y(x), y'(x))/y(x) \leqslant p(x, c_0, c_1/r(x))/c_0$ we get

$$1 \leqslant y(b_1)/y(x) + \alpha(x)r(x)y'(x)/y(x) + (1/c_0)\int_{b_1}^{x} \alpha(t) p(t, c_0\, c_1/r(t)) \mathrm{d}t. \tag{14}$$

Let ε be an arbitrary positive number, then by (13) there exists $b_2 \geqslant b_1$ so that $(1/c_0)\int_{b_2}^{\infty} \alpha(x) p(x, c_0, c_1/r(x)) \mathrm{d}x < \varepsilon$. Thus (14) implies

$$\liminf_{x\to\infty} \alpha(x)r(x)y'(x)/y(x) \geqslant 1 - \varepsilon. \tag{15}$$

Since for $x \geqslant b_2$ $(\alpha(x)r(x)y'(x) - y(x))' = -\alpha(x)p(x, y(x), y'(x)) < 0$, there exists a constant K such that $\alpha(x)r(x)y'(x)/y(x) \leqslant 1 + K/y(x)$. This implies $\limsup\limits_{x\to\infty} \alpha(x)r(x)y'(x)/y(x) \leqslant 1$. Since ε can be arbitrarily chosen we have from this inequality and (15)

$$\lim_{x\to\infty} \alpha(x)r(x)y'(x)/y(x) = 1. \tag{16}$$

Let $\varepsilon < 1/2$, then there exists a number $b \geqslant b_2$ such that for $x \geqslant b$, $\alpha(x)r(x)y'(x) \geqslant (1 - \varepsilon)y(x)$. By using this inequality, we get from (r)

$$(1 - \varepsilon)\{r(b)y'(b) - r(x)y'(x)\} = \int_b^x \alpha(t)r(t)y'(t)p(t, y(t), y'(t))/y(t)\mathrm{d}t \leqslant$$

$$\leqslant r(b)y'(b)(1/c_0)\int_b^{\infty} \alpha(t)p(t, c_0, c_1/r(t))\mathrm{d}t < r(b)y'(b)\varepsilon.$$

From this we have $0 < r(b)y'(b)(1 - 2\varepsilon)/(1 - \varepsilon) < r(x)y'(x)$ and because $r(x)y'(x)$ is non-increasing, $\lim\limits_{x\to\infty} r(x)y'(x) = c > 0$. From this and (16) we obtain $y(x) \sim c\alpha(x)$, which proves the theorem.

Theorem 7. *Let the function $p(x, u, v) > 0$ be continuous in D_5 and non-decreasing in v. Let there exist a number $\bar{c} > 0$ such that for $u \in \langle \bar{c}, \infty)$ $p(x, u, v)/u$ is non-decreasing in u for every fixed x and v. Let further for every number $c_1 > 0$ and $c_0 > \bar{c}$ $\int^{\infty} p(x, c_0\alpha(x), c_1/r(x))\mathrm{d}x < \infty$; then all solutions of* (r) *are either bounded or of the form $y(x) \sim c\alpha(x)$, $(c > \bar{c})$.*

The proof of this theorem is similar to that of Theorem 6 and can be therefore omitted.

Remark 3. It is evident from the proof of Theorem 6 that the assertion is valid also in the case when for every point of the region $D_6 = \{a \leqslant x < \infty, -\infty < u \leqslant 0, -\infty < v \leqslant 0\}$, $p(x, u, v) < 0$ and $p(x, u, v)/u$ is non-decreasing in u and (13) converges under suitable constants. In a similar way we can formulate also the assertions of Theorems 5 and 7 in D_6.

So far we have investigated only bounded solutions of (r) and solutions of the form $y(x) \sim c\alpha(x)$. But there can exist also other solutions of (r), e. g. the equation

$$(x^{1/2}y')' + (2x^{3/2}\ln^n x)^{-1}y^n = 0,\ n \geqslant 0 \tag{17}$$

has the solution $y(x) = \ln x$. In the case of $n = 1$, equation (17) has neither a nonoscillatory bounded solution nor a solution of the form $y(x) \sim c\alpha(x)$, which is evident from Theorems 1 and 3. In the case of $n > 1$, by Theorem 1, there exists a bounded solution of (17). In the case $0 \leqslant n < 1$, by Theorem 3, there exists a solution $y(x) \sim c\alpha(x)$ of (17). This leads to the following assertions.

Theorem 8. *Let there exist a function $f_2(x, u, v)$ such that for every point of D_5, $0 < p(x, u, v) \leqslant f_2(x, u, v)$. Let there further exist a positive function $F(u)$, continuous in the interval $\langle u_2, \infty)$ $(u_2 \geqslant 0)$, non-decreasing and such that $\int_{u_2}^{\infty} 1/F(u)\mathrm{d}u < \infty$. Let for some positive numbers c_0, c_1, all $u \in \langle u_2, \infty)$, $v \geqslant 0$ and all sufficiently large x be*

$$p(x, u, v)/F(u) \geqslant K_1 f_2(x, c_0, c_1/r(x))$$

$(K_1 > 0$ is a suitable constant); then if there exists an unboundea solution of (r), *there exists also a bounded solution of* (r).

Proof. Let $y(x)$ be an unbounded solution of (r), then there exists a number b such that for $x \geqslant b$, $y(x) \geqslant u_2$. Using this fact from (r) we get

$$y'(x)/F(y(x)) \geqslant (1/r(x))\int_x^\infty p(t, y(t), y'(t))/F(y(t))\mathrm{d}t \geqslant$$
$$\geqslant (K_1/r(x))\int_x^\infty f_2(t, c_0, c_1/r(t))\mathrm{d}t.$$

Integrating this inequality over the interval $\langle b, x\rangle$ we obtain

$$\int_{u_2}^\infty 1/F(u)\mathrm{d}u \geqslant K_1\int_b^x (1/r(t))\int_t^\infty f_2(v, c_0, c_1/r(v))\mathrm{d}v\mathrm{d}t \geqslant$$
$$\geqslant K_1\int_b^x \{\alpha(t) - \alpha(b)\}f_2(t, c_0, c_1/r(t))\mathrm{d}t.$$

This implies convergence of the integral on the right hand side from which, by Theorem 2, our assertion follows.

Theorem 9. *Let there exist a function $f_2(x, u, v)$ such that in D_5, $0 < p(x, u, v) \leqslant \leqslant f_2(x, u, v)$. Let there further exist a continuous function $G(s)$ having in the interval $\langle u_2, \infty)$ $(u_2 \geqslant 0)$ the following properties $G(s) > 0$, $\int_{u_2}^\infty G(s)/s^2\mathrm{d}s < \infty$ and $f_2(x, su, c_1/r(x)) \leqslant K_2G(s)(x, u, v)$ for all $u \geqslant 0, v \geqslant 0, s \in \langle u_2, \infty)$ and all sufficiently large x, whwere c_1 and K_2 are suitable constants.*

Then if equation (r) *has some solution, it has also a solution of the form $y(x) \sim c\alpha(x)$.*

Proof. Let $y(x)$ be a solution of (r), then there exist constants b and $c_2 > 0$ such that for $x \geqslant b$ from (r) we get

$$y(x) \geqslant c_2\alpha(x)\int_x^\infty p(t, y(t), y'(t))\mathrm{d}t.$$

By the properties of the functions $f_2(x, u, v)$, $p(x, u, v)$ and the last inequality we obtain

$$(18)\qquad f_2(x, c_0\alpha(x), c_1/r(x)) \leqslant f_2(x, c_3y(x)\{\int_x^\infty p(t, y(t), y'(t))\mathrm{d}t\}^{-1}, c_1/r(x)) \leqslant$$
$$\leqslant K_2G(c_3\{\int_x^\infty p(t, y(t), y'(t))\mathrm{d}t\}^{-1})p(x, y(x), x'(y)),$$

where c_0 is a positive constant such that $c_0 \leqslant c_1$. Integrating (18) we have

$$\int_{x_0}^x f_2(t, c_0\alpha(t), c_1/r(t))\,\mathrm{d}t \leqslant K_2\int_{x_0}^x G(c_3\{\int_t^\infty p(v, y(v), y'(v))\mathrm{d}v\}^{-1})p(t, y(t), y'(t))\mathrm{d}t \leqslant$$
$$\leqslant K_2c_3\int_{u_2}^\infty G(s)/s_2\mathrm{d}s < \infty.$$

This implies the convergence of the integral on the left hand side and thus, by Theorem 4, equation (r) has the solution $y(x) \sim c\alpha(x)$.

Remark 4. The assertions of Theorems 8 and 9 will be valid also in D_6. It is sufficient only to suppose that $f_2(x, u, v) \leqslant p(x, u, v) < 0$ for all points of D_6, the functions $F(u)$ and $G(s)$ are negative in the interval $(-\infty, u_2\rangle$ $(u_2 \leqslant 0)$ and c_0, c_1, K_1, K_2 are suitable constants. This follows from Remarks 1 and 2.

Remark 5. In case when the function $p(x, u, v)$ satisfies condition 3) and all assumptions of Theorems 5, 6, 7, 8 and 9 are valid, the assertions of these theorems will be also valid in D. It must be noted that this concerns only nonoscillatory solutions.

II.

Theorem 10. *Let there exist two functions $f_1(x, u, v)$ and $f_2(x, u, v)$ such tha for every point $(x, u, v) \in D$, $u \neq 0$ we have*

$$0 < uf_1(x, u, v) \leqslant up(x, u, v) \leqslant uf_2(x, u, v). \tag{19}$$

Besides, let for every continuously differentiable monotone function $\beta(x)$ such that $\lim_{x\to\infty} |\beta(x)| = \infty$ be $\int^{\infty} \beta^{-1}(x) f_1(x, \beta(x), \beta'(x))\mathrm{d}x = \infty$. Then.

1. *If for eveyr constant $c_0 \neq 0$, $\int^{\infty} \alpha(x)|f_1(x, c_0, 0)|\mathrm{d}x = \infty$, all solutions of* (r) *are oscillatory.*

2. *If all solutions of* (r) *are oscillatory, then for every positiev (negative) numbers c_0, c_1 we have $\int^{\infty} \alpha(x)|f_2(x, c_0, c_1/r(x))|\mathrm{d}x = \infty$.*

Proof. 1. Let us suppose that $y(x)$ is a nonoscillatory solution of (r). Without loss of generality we can suppose that for $x \geqslant b \geqslant a$, $y(x) > 0$. Then $y(x)$ is an increasing function and $r(x)y'(x)$ a decreasing one. The function $y(x)$ is unbounded. If it were not so, there would exist two numbers c_0 and c such that for $x \geqslant b$, $c \geqslant y(x) \geqslant c_0$. Using this fact and integrating equation (r) successively over the intervals $\langle t, x\rangle$, $\langle b, x\rangle$ $(b < t < x)$ we have

$$c \geqslant y(x) \geqslant \int_b^x \{\alpha(t) - \alpha(b)\} p(t, y(t), y'(t))\mathrm{d}t \geqslant \int_b^x \{\alpha(t) - \alpha(b)\} f_1(t, c_0, 0)\mathrm{d}t.$$

The last inequality holds for every $x \geqslant b$ and this implies $\int^{\infty} \alpha(x) f_1(x, c_0, 0)\mathrm{d}x < \infty$. But this contradicts the assumption of the theorem and thus $y(x)$ in an unbounded solution of (r). Since we have

$$(r(x)y'(x)/y(x))' = -p(x, y(x), y'(x))/y(x) - r(x)y'^2(x)/y^2(x),$$

integrating this equality over $\langle b, x\rangle$ we obtain

$$r(x)y'(x)/y(x) = r(b)y'(b)/y(b) - \int_b^x y^{-1}(t)p(t, y(t), y'(t))\mathrm{d}t -$$
$$- \int_b^x r(t)y'^2(t)y^{-2}(t)\mathrm{d}t.$$

The function $r(x)y'(x)/y(x)$ is bounded, because $r(x)y'(x) > 0$ and decreases for $x \geqslant b$ and $y(x)$ is an unbounded solution of (r). This implies that

$$\int^\infty y^{-1}(x)p(x, y(x), y'(x))\mathrm{d}x < \infty, \quad \text{i. e.} \quad \int^\infty y^{-1}(x)f_1(x, y(x), y'(x))\mathrm{d}x < \infty,$$

but this also contradicts the assumption of the theorem. Thus all solutions of (r) are oscillatory.

2. If there existed two positive numbers c_0 and c_1 (similarly negative ones) such that $\int^\infty \alpha(x)f_2(x, c_0, c_1/r(x))\mathrm{d}x < \infty$, then by Theorem 2 (Remark 1) there would exist a nonoscillatory solution of (r), which is a contradiction.

Theorem 11. *Let the functions $p(x, u, v)$, $f_1(x, u, v)$ and $f_2(x, u, v)$ in the region D satisfy* (19). *Let there exist a function $F(u)$ continuous, non-decreasing in the interval $(-\infty, \infty)$ and such that $uF(u) > 0$, $\int_\varepsilon^\infty 1/F(u)\mathrm{d}u < \infty$, $\int_{-\varepsilon}^{-\infty} 1/F(u)\mathrm{d}u < < \infty$ for every $\varepsilon > 0$. Let for some positive numbers c_0, c_1, K_1, all $u > 0$, $v \geqslant 0$ (some negative numbers c_0, c_1, K_1, all $u < 0$, $v \leqslant 0$) and all sufficiently large x be $f_1(x, u, v)/F(u) \geqslant K_1 f_2(x, c_0, c_1/r(x))$.*

Then a necessary and sufficient condition in order that all solutions of (r) *be oscillatory is*

$$\int^\infty \alpha(x)|f_2(x, c_0, c_1/r(x))|\mathrm{d}x = \infty. \tag{20}$$

Proof. 1. Let $y(x)$ be a nonoscillatory solution of (r). Without loss of generality it can be supposed that for $x \geqslant b$, $y(x) \geqslant \varepsilon$, where ε is some positive number. Using the properties of f_1, p and f_2, from (r) we obtain for $x \geqslant b$

$$y'(x)/F(y(x)) \geqslant (r(x)F(y(x)))^{-1}\int_x^\infty p(t, y(t), y'(t))\mathrm{d}t \geqslant \tag{21}$$
$$\geqslant r^{-1}(x)\int_x^\infty F^{-1}(y(t))f_1(t, y(t), y'(t))\mathrm{d}t \geqslant K_1 r^{-1}(x)\int_x^\infty f_2(t, c_0, c_1/r(t))\mathrm{d}t.$$

Integrating this inequality over the interval $\langle b, x\rangle$ we have

$$\int_\varepsilon^\infty 1/F(u)\mathrm{d}u \geqslant K_1\int_b^x r^{-1}(t)\int_t^\infty f_2(\nu, c_0, c_1/r(\nu))\mathrm{d}\nu\,\mathrm{d}t \geqslant$$
$$\geqslant K_1\int_b^x \{\alpha(t) - \alpha(b)\}f_2(t, c_0, c_1/r(t))\mathrm{d}t.$$

This implies convergence of the last integral, hence we get a contradiction to (20). Thus the sufficient condition is proved.

2. Let all solutions of (r) be oscillatory and let $\int^{\infty} \alpha(x)|f_2(x, c_0, c_1/r(x))|\mathrm{d}x < \infty$. Then by Theorem 2 or Remark 1 equation (r) has a nonoscillatory solution, but this again contradicts the assumption of the theorem and the proof is complete.

In the following theorem there will be given a sufficient condition in order that all solutions of (r) be oscillatory. This condition is similar to that of Theorem 11.

Theorem 12. *Let there exist a function $f_1(x, u, v)$ such that for every point $(x, u, v) \in D$, $u \neq 0$ we have*

$$0 < uf_1(x, u, v) \leqslant up(x, u, v). \tag{22}$$

Let there further the function $F(u)$ be continuous on the set $N = (-\infty, -u_2\rangle \cup \cup \langle u_2, \infty)$ ($u_2 > 0$ is a suitable number), non-decreasing and such that $uF(u) > 0$, $\int_{u_2}^{\infty} 1/F(u)\mathrm{d}u < \infty$, $\int_{-u_2}^{-\infty} 1/F(u)\mathrm{d}u < \infty$. Let for some positive numbers c_0, K_1, all $u \in \langle u_2, \infty)$, $v \geqslant 0$ (some negative numbers c_0, K_1, all $u \in (-\infty, -u_2\rangle$, $v \leqslant 0$) and all sufficiently large x be $f_1(x, u, 0)/F(u) \geqslant K_1 f_1(x, c_0, 0)$.

Then if $\int^{\infty} \alpha(x)|f_1(x, c, 0)|\mathrm{d}x = \infty$ for $0 < |c| \leqslant |c_0|$, all solutions of (r) *are oscillatory.*

Proof. Let $y(x)$ be a positive nonoscillatory solution of (r) (likewise for $y(x) < 0$). As in the first part of the proof of Theorem 10 we can similarly prove that $y(x)$ is an unbounded solution. Then for $x \geqslant b$, $y(x) \geqslant u_2$. Thus from (r), as in (21) we get

$$y'(x)/F(y(x)) \geqslant K_1 r^{-1}(x) \int_x^{\infty} f_1(t, c_0, 0)\mathrm{d}t.$$

Integration of this inequality over $\langle b, x\rangle$ yields

$$\int_{u_2}^{\infty} 1/F(u)\mathrm{d}u \geqslant K_1 \int_b^x \{\alpha(t) - \alpha(b)\} f_1(t, c_0, 0)\mathrm{d}t.$$

However, this contradicts the assumption and proves the theorem.

Theorems 11 and 12 generalize the known criteria of [1], [5]. The independence of the criteria from Theorems 10 and 12 is shown by the following examples

$$y''(x) + xy(x) \ln(2 + y^2(x)) = 0 \tag{23}$$

$$y''(x) + x^{-2}y^3(x) = 0. \tag{24}$$

All solutions of (24) are oscillatory by Theorem 12, because we can choose $F(u) = u^3$. But the conditions of Theorem 10 are not satisfied, because if we put e. g. $\beta(x) = x^{1/3}$, then $\int^{\infty} \beta^3(x)(x^2\beta(x))^{-1}\mathrm{d}x = \int^{\infty} x^{-4/3}\mathrm{d}x < \infty$. Similarly all solutions of (23) are oscillatory by Theorem 10, since for every function $\beta(x)$, $\int^{\infty} x \ln(2 + \beta^2(x))\mathrm{d}x = \infty$, but the conditions of Theorem 12 are not satisfied.

Theorem 13. *Let there exist two function $f_1(x, u, v)$ and $f_2(x, u, v)$ satisfying* (19) *on D. Let the function $G(s)$ be continuous on the set $N = (-\infty, -u_2\rangle \cup \cup \langle u_2, \infty)$ ($u_2 > 0$ is a suitable number) having these properties $sG(s) > 0$, $\int_{u_2}^{\infty} G(s)/s^2\mathrm{d}s < \infty$, $\int_{-u_2}^{-\infty} G(s)/s^2\mathrm{d}s < \infty$ and*

$$f_2(x, su, c_1/r(x)) \leqslant K_2G(s)f_1(x, u, v) \tag{25}$$

both for all $u > 0$, $v \geqslant 0$, $s \in \langle u_2, \infty)$ or $u < 0$, $v \leqslant 0$, $s \in \langle -\infty, -u_2\rangle$ and sufficiently large x, where $c_1 > 0$, $K_2 > 0$.

Then all solutions of the equation (r) *are oscillatory if and only if $\int^{\infty} f_2(x, c_0\alpha(x), c_1/r(x))\mathrm{d}x = \infty$ for all $0 < c_0 \leqslant c_1$.*

Proof. 1. Let $y(x)$ be a nonoscillatory solution of (r). Without loss of generality we can suppose $y(x) > 0$. Then there exist positive numbers b and $c_0 \leqslant c_1$ such that for $x \geqslant b$ from (r) we get $y(x) \geqslant c_0\alpha(x)\int_x^{\infty} p(t, y(t), y'(t))\mathrm{d}t$, where $\{\int_x^{\infty} p(t, y(t), y'(t)\mathrm{d}t\}^{-1} \geqslant u_2$. By (19), (25) and by the preceding inequalities we have

$$\begin{aligned} f_2(x, c_0\alpha(x), c_1/r(x)) &\leqslant f_2(x, y(x)\{\int_x^{\infty} p(t, y(t), y'(t))\mathrm{d}t\}^{-1}, c_1/r(x)) \leqslant \\ &\leqslant K_2G(\{\int_x^{\infty} p(t, y(t), y'(t))\mathrm{d}t\}^{-1})f_1(x, y(x), y'(x)) \leqslant \\ &\leqslant KG(\{\int_x^{\infty} p(t, y(t), y'(t))\mathrm{d}t\}^{-1})p(x, y(x), y'(x)). \end{aligned}$$

Integrating this over the interval $\langle b, x\rangle$ we obtain

$$\int_b^x f_2(t, c_0\alpha(t), c_1/r(t))\mathrm{d}t \leqslant K_2\int_{u_2}^{\infty} G(s)/s^2\mathrm{d}s < \infty.$$

This implies convergence of the integral on the left hand side of the last inequality and contradicts the assumption.

2. Let all solutions of (r) be oscillatory and let $\int\limits^{\infty} f_2(x, c_0\alpha(x), c_1/r(x))\mathrm{d}x < \infty$ for some $0 < c_0 \leqslant c_1$. Then by Theorem 4 equation (r) has a nonoscillatory solution and thus we get a contradiction. This proves the theorem.

Theorem 14. *Let the functions $p(x, u, v)$ and $f_1(x, u, v)$ satisfy* (22) *in D. Let there exist a function $G(s)$ with the properties as in Theorem* 13 *such that $f_1(x, su, 0) \leqslant K_2G(s)f_1(x, u, 0)$ both for all $u > 0$, $s \in \langle u_2, \infty)$ or $u < 0$, $s \in (-\infty, -u_2\rangle$ and all sufficiently large x, where $K_2 > 0$. If there exists $c > 0$ such that for every $0 < c_0 \leqslant c$, $\int\limits^{\infty} f_1(x, c_0\alpha(x), 0)\mathrm{d}x = \infty$, then all solutions of* (r) *are oscillatory.*

The proof is analogous to that of the first part of Theorem 13 and therefore can be omitted.

The last two theorems generalize some assertion from [2] and [6].

The next theorem is a generalization of a criterion given in [10] for a linear differential equation of the second order.

Theorem 15. *Let the function $p(x, u, v)$ satisfy condition* 3) *in D, let it further be non-decreasing in v and such that $p(x, u, 0)/u$ is non-decreasing for $u \in (-\infty, 0)$, non-increasing for $u \in (0, \infty)$ and an arbitrary fixed x. Let there exists a positive function $w(x) \in C^1 \langle a, \infty)$ satisfying*

$$\int\limits^{\infty} r(x)w'^2(x)w^{-1}(x)\mathrm{d}x < \infty. \tag{26}$$

Besides, let for any $c \neq 0$

$$\int\limits^{\infty} w(x)\alpha^{-1}(x)|p(x, c\alpha(x), 0)|\mathrm{d}x = \infty; \tag{27}$$

then every solution of (r) *is oscillatory.*

Proof. Let $y(x)$ be a nonoscillatory solution of (r) so that $y(x) > 0$ for $x \geqslant b \geqslant a$. From (r) we get

$$w(x)(r(x)y'(x)/y(x))' =$$
$$= -w(x)p(x, y(x), y'(x))/y(x) - w(x)r(x)(y'(x)/y(x))^2.$$

By integration over the interval $\langle b, x\rangle$ we have

$$w(x)r(x)y'(x)/y(x) = w(b)r(b)y'(b)/y(b) + \int\limits_b^x w'(t)r(t)y'(t)/y(t)\mathrm{d}t - \tag{28}$$
$$- \int\limits_b^x w(t)p(t, y(t), y'(t))/y(t)\mathrm{d}t - \int\limits_b^x w(t)r(t)(y'(t)/y(t))^2\mathrm{d}t.$$

Since the function $r(x)y'(x) > 0$ is decreasing, there exists a constant c such that $y(x) \leqslant c\alpha(x)$ for every $x \geqslant b$. Thus by assumption the following is true

$$(29) \qquad p(x, y(x), y'(x))/y(x) \geqslant p(x, c\alpha(x), 0)/c\alpha(x).$$

Using in (28) the Cauchy inequality and (29) we have

$$(30) \qquad w(x)r(x)y'(x)/y(x) \leqslant w(b)r(b)y'(b)/y(b) + \\ + [\int_b^x r(t)w'^2(t)/w(t)\mathrm{d}t]^{1/2} \cdot [\int_b^x w(t)r(t)(y'(t)/y(t))^2\mathrm{d}t]^{1/2} - \\ - c^{-1}\int_b^x w(t)\alpha^{-1}(t)p(t, c\alpha(t), 0)\mathrm{d}t - \int_b^x w(t)r(t)(y'(t)/y(t))^2\mathrm{d}t.$$

By (26) we can take number b such that $\int_b^\infty r(t)w'^2(t)w^{-1}(t)\mathrm{d}t < 1$. Then by (27) and (30) we get $w(x)r(x)y'(x)/y(x) < 0$ for all sufficiently large x. Thus we get a contradiction. For $y(x) < 0$ the consideration is similar.

The following theorem deals with the increase or decrease of the „amplitudes" of oscillatory solutions. Here and in the next theorems $\alpha(x)$ can be bounded.

Theorem 16. *Let condition* 3) *hold in D. Denote by b, c the successive zeros of some solution $y(x)$ of* (r), *by b', c' the successive zeros of $y'(x)$. Then the following assertions hold:*

a) *Let the function $r(x)p(x, u, v)$ be non-increasing in x for $u > 0$, non-decreasing for $u < 0$ and all v. Let it further be non-decreasing in v for every fixed x and u, then $r(b)|y'(b)| \geqslant r(c)|y'(c)|$.*

b) *Let the function $r(x)p(x, u, v)$ be non-decreasing in x for $u > 0$, non-increasing for $u < 0$ and all v. Let it further be non-increasing in v for every fixed x and u, then $r(b)|y'(b)| \leqslant r(c)|y'(c)|$.*

Besides, if $p(x, u, v)$ is odd in u, then in the case a) $|y(b')| \leqslant |y(c')|$ *and in* b) $|y(b')| \geqslant |y(c')|$.

Proof. Without loss of generality we can suppose that $y(x) > 0$ for $x \in (b, c)$. From (r) we see that in the interval (b, c) there lies one and only one zero of $y'(x)$, denote it by b'. In the interval (b', c') there lies one and only one zero of $y(x)$, denote it by c. Let us multiply (r) by $r(x)y'(x)$ and integrate over $\langle b, b'\rangle$ or $\langle b', c\rangle$ then we get

$$(r(b)y'(b))^2 = 2\int_b^{b'} r(x)p(x, y(x), y'(x))y'(x)\mathrm{d}x \geqslant 2\int_0^{y(b')} r(x)p(x, s, 0)\mathrm{d}s,$$

$$(r(c)y'(c))^2 = -2\int_{b'}^{c} r(t)p(t, y(t), y'(t))y'(t)\mathrm{d}t \leqslant 2\int_0^{y(b')} r(t)p(t, s, 0)\mathrm{d}s.$$

In the case a) $r(x)p(x, s, 0) \geqslant r(t)p(t, s, 0)$, hence we have $r(b)|y'(b)| \geqslant$ $\geqslant r(c)|y'(c)|$.

Denote $\nu = \min_{x \in \langle b', c'\rangle} y'(x)$ and suppose that $p(x, u, v)$ is odd in u. Then from (r) we obtain

$$(r(c)y'(c))^2 = -2\int_{b'}^{c} r(x)p(x, y(x), y'(x))y'(x)\mathrm{d}x \geqslant 2\int_{0}^{y(b')} r(x)p(x, s, v)\mathrm{d}s,$$

$$(r(c)y'(c))^2 = 2\int_{c}^{c'} r(t)p(t, y(t), y'(t))y'(t)\mathrm{d}t \leqslant 2\int_{0}^{|y(c')|} r(t)p(t, s, v)\mathrm{d}s.$$

This implies that $\int_0^{|y(c')|} r(t)p(t, s, v)\mathrm{d}s \geqslant \int_0^{|y(b')|} r(x)p(x, s, v)\mathrm{d}s$. Since for $t > x$ we have $r(x)p(x, s, v) \geqslant r(t)p(t, s, v)$, the preceding inequality is possible only if $|y(c')| \geqslant |y(b')|$.

The case b) can be proved analogously.

Remark 6. Denote by $\{x_n\}$ the sequence of successive zeros of some oscillatory solution $y(x)$ of (r), $\{x_n'\}$ zeros of $y'(x)$. Then by assumptions of Theorem 16 in the case a) the sequence $\{r(x_n)|y'(x_n)|\}$ is non-increasing and $\{|y(x_n')|\}$ is non-decreasing. Besides, the function $r(x)y'(x)$ has extrema in x_n, therefore there exists a number K such that $|y(x)| \leqslant K\alpha(x)$. If $\alpha(x)$ is bounded, then every solution of (r) is bounded, which follows from the last inequality and Theorem 5.

Remark 7. Let $y(x) \not\equiv 0$ be an oscillatory solution of (r) and the assumptions of Theorem 16 be satisfied. Then the sequence of zeros of $y(x)$ has a cluster point only at infinity. If it were not so there would exist a finite cluster point $\bar{x}$ such that by the continuity of $y(x)$ and $y'(x)$ we shold have $y(\bar{x}) = y'(\bar{x}) = 0$. Hence we get a contradiction in both cases a) and b).

Theorem 17. *Let the function $p(x, u, v)$ be non-decreasing in u, v for every fixed x and* 3) *hold in D. Besides, let* a) *from Theorem* 16 *hold and $p(x, u, v)/u$ be even and non-decreasing in u for $u > 0$.*

If for any positive number c, $\int^{\infty} p(x, c\alpha(x), c/r(x))\mathrm{d}x < \infty$, (r) *has no oscillatory solution, besides a trivial one.*

Proof. Let $y(x) \not\equiv 0$ be an oscillatory solution of (r). Let $y(x_n) = y'(x_n') = 0$ and $y(x) > 0$ for $x \in (x_n, x_n'\rangle$. By Theorem 16 the sequence $\{r(x_n)|y'(x_n)|\}$ is non-increasing, thus there exists a number c such that $r(x)y'(x) \leqslant c$ for all $x \geqslant x_n$. From this we have $y(x) \leqslant c\alpha(x)$, $y'(x) \leqslant c/r(x)$ for $x \geqslant x_n$. By assumptions of the Theorem and the last inequalities we obtain

$$p(x, y(x), c/r(x))(r(x_n)y'(x_n))^{-1} \leqslant p(x, c\alpha(x), c/r(x))/c. \tag{31}$$

Integrating (r) over the interval $\langle x_n, x_n'\rangle$ we get

$$r(x_n)y'(x_n) = \int_{x_n}^{x_n'} p(x, y(x), y'(x))\mathrm{d}x \leqslant \int_{x_n}^{x_n'} p(x, y(x), c/r(x))\mathrm{d}x.$$

Hence and by (31) we have

$$1 \leqslant \int_{x_n}^{x_n'} p(x, y(x), c/r(x))(r(x_n)y'(x_n))^{-1}\mathrm{d}x \leqslant \tag{32}$$
$$\leqslant c^{-1} \int_{x_n}^{\infty} p(x, c\alpha(x), c/r(x))\mathrm{d}x.$$

By Remark 7 the set of zeros of $y(x)$ cannot have a finite cluster point, therefore it is possible to find x_n such that $c^{-1} \int_{x_n}^{\infty} p(x, c\alpha(x), c/r(x))\mathrm{d}x < 1$. Hence we get a contradiction to (32) and the proof is complete.

Theorem 18. *Let the function $p(x, u, v)$ be such that the assumptions of Theorem 17 hold, where $p(x, u, v)/u$ is non-increasing in u for $u > 0$. If for every $c_1 > 0$ and all small positive numbers c_0 $\int^{\infty} \alpha(x)p(x, c_0, c_1/r(x))\mathrm{d}x < \infty$, then* (r) *has no oscillatory solution, besides a trivial one.*

Proof. Let $y(x) \not\equiv 0$ be an oscillatory solution of (r) such that $y(x) > 0$ for $x \in \langle x_n, x_n' \rangle$. By Theorem 16 the sequence $\{|y(x_n')|\}$ is non-decreasing and $\{r(x_n)|y'(x_n)|\}$ non-increasing. This implies the existence of two numbers $c_0 > 0$, $c_1 > 0$ such that $c_0 < |y(x_n')|$ (c_0 can be chosen small) and $y'(x) \leqslant \leqslant c_1/r(x)$ for all $x \geqslant x_n$. Integrating (r) over the intervals $\langle x, x_n' \rangle$, $\langle x_n, x_n' \rangle$ and using the assumptions we get

$$1 \leqslant \int_{x_n}^{x_n'} y^{-1}(x_n')\alpha(x)p(x, y(x), y'(x))\mathrm{d}x \leqslant c_0^{-1} \int_{x_n}^{\infty} \alpha(x)p(x, c_0, c_1/r(x))\mathrm{d}x.$$

Using the same consideration as at the end of the proof of Theorem 17, we can easily complete our proof.

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