Josef Kaucký

SomeRremarks on a Combinatorial Problem

Matematický časopis, Vol. 22 (1972), No. 3, 215--218

Persistent URL: http://dml.cz/dmlcz/126523

Matematický časopis 22 (1972), No. 3

SOME REMARKS ON A COMBINATORIAL PROBLEM

JOSEF KAUCKÝ, Bratislava

1.

In Mathematics Magazine 42 (1969), 154—156, there was published with a proof the following combinatorial identity

$$\bar{S}_\gamma = D_\gamma, \tag{1}$$

in which

$$\bar{S}_\gamma = \sum_{j=0}^{\gamma-1} \sum_{i=0}^{j} i!(j-i)! \tag{2}$$

and

$$D_\gamma = \begin{vmatrix} 0! & -\binom{1}{1} & 0 & \dots & 0 \\ 1! & +\binom{2}{1} & -\binom{2}{2} & \dots & 0 \\ 2! & -\binom{3}{1} & +\binom{3}{2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ (\gamma-2)! & (-1)^{\gamma-1}\binom{\gamma-1}{1} & (-1)^{\gamma-2}\binom{\gamma-1}{2} & \dots & -\binom{\gamma-1}{\gamma-1} \\ (\gamma-1)! & (-1)^{\gamma}\binom{\gamma}{1} & (-1)^{\gamma-1}\binom{\gamma}{2} & \dots & +\binom{\gamma}{\gamma-1} \end{vmatrix} \tag{3}$$

The proof is based on the recurrent relation

$$S_0 = 1, \; S_j = jS_{j-1} + S_0, \tag{4}$$

where

$$S_j = \sum_{i=0}^{j} i!(j-i)! \tag{5}$$

Now a simple numerical calculation gives the values

$$D_1 = 1,\quad D_2 = 3,\quad D_3 = 8,\quad D_4 = 24,\quad D_5 = \underline{89},\quad \ldots$$

$$\bar{S}_1 = 1,\quad \bar{S}_2 = 3,\quad \bar{S}_3 = 8,\quad \bar{S}_4 = 24,\quad \bar{S}_5 = \underline{88},\quad \ldots$$

$$S_0 = 1,\quad S_1 = 2,\quad S_2 = 5,\quad S_3 = 16,\quad S_4 = \underline{64},\quad \ldots$$

$$1 + jS_{j-1} = 2, 5, 16, \underline{65}, \ldots \text{ for } j = 1, 2, 3, 4, \ldots,$$

which imply the fact that neither the identity (1) nor the recurrent relation (4) are correct the less so the proof.

The following paragraphs contain the proof of the correct recurrent relation

$$S_j = j! + \frac{j+1}{2} S_{j-1} \tag{6}$$

for the sums S_j and the proof of the new relation

$$D_\gamma = \sum_{j=1}^{\gamma} \binom{\gamma}{j} (j-1)! \tag{7}$$

for the determinants D_γ.

2.

To be able to give a proof of the recurrent relation (6), let us put

$$S_j = \sum_{i=0}^{j} i!(j-i)! =$$

$$= j! + \sum_{i=1}^{j} i!(j-i)! = j! + \sum_{i=0}^{j-1} (1+i)i!(j-1-i)! =$$

$$= j! + S_{j-1} + S'_{j-1},$$

where

$$S'_{j-1} = \sum_{i=0}^{j-1} ii!(j-1-i)! = \sum_{i=0}^{j-1} (j-1-i)\,(j-1-i)!i! =$$

$$= (j-1)S_{j-1} - S'_{j-1},$$

so that

$$S'_{j-1} = \frac{j-1}{2} S_{j-1}.$$

Thus we get

$$S_j = j! + \frac{j+1}{2} S_{j-1}.$$

Using this relation we prove that

$$S_j = \frac{(j+1)!}{2^{j+1}} \left(2 + \frac{2^2}{2} + \ldots + \frac{2^{j+1}}{j+1} \right). \tag{8}$$

The formula is evidently valid for $j = 1$ and from the validity of (8) and from the equation (6) we have

$$S_{j+1} = (j+1)! + \frac{(j+2)!}{2^{j+2}} \left(2 + \frac{2^2}{2} + \ldots + \frac{2^{j+1}}{j+1} \right) =$$
$$= \frac{(j+2)!}{j+2} \left(2 + \frac{2^2}{2} + \ldots + \frac{2^{j+1}}{j+1} + \frac{2^{j+2}}{j+2} \right),$$

so that (8) is valid even with $(j+1)$ instead of j and therefore it is valid for each j.

Let us point out in addition that

$$S_j = \sum_{i=0}^{j} i!(j-i)! = j! \sum_{i=0}^{j} \frac{1}{\binom{j}{i}},$$

so that from (6) there results the well known formula

$$\sum_{i=0}^{j} \frac{1}{\binom{j}{i}} = \frac{j+1}{2^{j+1}} \left(2 + \frac{2^2}{2} + \ldots + \frac{2^{j+1}}{j+1} \right). \tag{9}$$

3.

To be able to prove the relation (7) let us take into account that through a successive developement of the determinant according to the elements of the last column we easily get the relation

$$D_\gamma = (\gamma - 1)! + \sum_{j=1}^{\gamma-1} (-1)^{j-1} \binom{\gamma}{j} D_{\gamma-j}, \tag{10}$$

so that we may use the mathematical induction. For $\gamma = 1$ the relation (7) evidently holds. Let us therefore assume that it holds for the indices 1 to $(\gamma - 1)$ and let us calculate

$$D_\gamma = (\gamma - 1)! + (-1)^{\gamma-1} \sum_{j=1}^{\gamma-1} (-1)^j \binom{\gamma}{j} D_j = (\gamma - 1)! +$$

$$+(-1)^{\gamma-1} \sum_{j=1}^{\gamma-1} (-1)^j \binom{\gamma}{j} \sum_{i=1}^{j} \binom{j}{i} (i-1)! =$$

$$= (\gamma - 1)! + (-1)^{\gamma-1} \sum_{i=1}^{\gamma-1} (i-1)! \sum_{j=i}^{\gamma-1} (-1)^j \binom{j}{i} \binom{\gamma}{j}.$$

Now the well known formula

$$\binom{n+\mu}{n} \binom{x}{n+\mu} = \binom{x}{n} \binom{x-n}{\mu}$$

gives

$$D_\gamma = (\gamma - 1)! + (-1)^{\gamma-1} \sum_{i=1}^{\gamma-1} \binom{\gamma}{i} (i-1)! \sum_{j=i}^{\gamma-1} (-1)^j \binom{\gamma - i}{j - i} =$$

$$= (\gamma - 1)! + (-1)^{\gamma-1} \sum_{i=1}^{\gamma-1} (-1)^i \binom{\gamma}{i} (i-1)! \sum_{j=0}^{\gamma-i-1} (-1)^j \binom{\gamma - i}{j},$$

and with the help of another relation

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k} = \binom{n - x}{n}$$

we further get

$$D_\gamma = (\gamma - 1)! + \sum_{i=1}^{\gamma-1} \binom{\gamma}{i} (i-1)! \, (-1)^{\gamma-i-1} \binom{-1}{\gamma - i - 1}.$$

And as

$$(-1)^n \binom{-1}{n} = 1,$$

we finally obtain

$$D_\gamma = (\gamma - 1)! + \sum_{i=1}^{\gamma-1} \binom{\gamma}{i} (i-1)! = \sum_{j=1}^{\gamma} \binom{\gamma}{j} (j-1)!,$$

which is the required relation.