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SOME REMARKS ON A COMBINATORIAL PROBLEM

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1.

In Mathematics Magazine 42 (1969), 154–156, there was published with a proof the following combinatorial identity

$$(1) \quad \bar{S}_\gamma = D_\gamma,$$

in which

$$(2) \quad \bar{S}_\gamma = \sum_{j=0}^{\gamma-1} \sum_{i=0}^j i!(j-i)!$$

and

$$(3) D_\gamma = \begin{vmatrix} 0! & -\binom{1}{1} & 0 & \dots & 0 \\ 1! & +\binom{2}{1} & -\binom{2}{2} & \dots & 0 \\ 2! & -\binom{3}{1} & +\binom{3}{2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ (\gamma-2)! & (-1)^{\gamma-1} \binom{\gamma-1}{1} & (-1)^{\gamma-2} \binom{\gamma-1}{2} & \dots & -\binom{\gamma-1}{\gamma-1} \\ (\gamma-1)! & (-1)^\gamma \binom{\gamma}{1} & (-1)^{\gamma-1} \binom{\gamma}{2} & \dots & +\binom{\gamma}{\gamma-1} \end{vmatrix}$$

The proof is based on the recurrent relation

$$(4) \quad S_0 = 1, S_j = jS_{j-1} + S_0,$$

where

$$(5) \quad S_j = \sum_{i=0}^j i!(j-i)!$$

Now a simple numerical calculation gives the values

$$\begin{aligned} D_1 &= 1, & D_2 &= 3, & D_3 &= 8, & D_4 &= 24, & D_5 &= \underline{89}, & \dots \\ \bar{S}_1 &= 1, & \bar{S}_2 &= 3, & \bar{S}_3 &= 8, & \bar{S}_4 &= 24, & \bar{S}_5 &= \underline{88}, & \dots \\ S_0 &= 1, & S_1 &= 2, & S_2 &= 5, & S_3 &= 16, & S_4 &= \underline{64}, & \dots \\ 1 + jS_{j-1} &= 2, 5, 16, \underline{65}, \dots & \text{for } j &= 1, 2, 3, 4, \dots, \end{aligned}$$

which imply the fact that neither the identity (1) nor the recurrent relation (4) are correct the less so the proof.

The following paragraphs contain the proof of the correct recurrent relation

$$(6) \quad S_j = j! + \frac{j+1}{2} S_{j-1}$$

for the sums  $S_j$  and the proof of the new relation

$$(7) \quad D_\gamma = \sum_{j=1}^{\gamma} \binom{\gamma}{j} (j-1)!$$

for the determinants  $D_\gamma$ .

## 2.

To be able to give a proof of the recurrent relation (6), let us put

$$\begin{aligned} S_j &= \sum_{i=0}^j i!(j-i)! = \\ &= j! + \sum_{i=1}^j i!(j-i)! = j! + \sum_{i=0}^{j-1} (1+i)i!(j-1-i)! = \\ &= j! + S_{j-1} + S'_{j-1}, \end{aligned}$$

where

$$\begin{aligned} S'_{j-1} &= \sum_{i=0}^{j-1} i!(j-1-i)! = \sum_{i=0}^{j-1} (j-1-i)(j-1-i)! = \\ &= (j-1)S_{j-1} - S'_{j-1}, \end{aligned}$$

so that

$$S'_{j-1} = \frac{j-1}{2} S_{j-1}.$$

Thus we get

$$S_j = j! + \frac{j+1}{2} S_{j-1}.$$

Using this relation we prove that

$$(8) \quad S_j = \frac{(j+1)!}{2^{j+1}} \left( 2 + \frac{2^2}{2} + \dots + \frac{2^{j+1}}{j+1} \right).$$

The formula is evidently valid for  $j = 1$  and from the validity of (8) and from the equation (6) we have

$$\begin{aligned} S_{j+1} &= (j+1)! + \frac{(j+2)!}{2^{j+2}} \left( 2 + \frac{2^2}{2} + \dots + \frac{2^{j+1}}{j+1} \right) = \\ &= \frac{(j+2)!}{j+2} \left( 2 + \frac{2^2}{2} + \dots + \frac{2^{j+1}}{j+1} + \frac{2^{j+2}}{j+2} \right), \end{aligned}$$

so that (8) is valid even with  $(j+1)$  instead of  $j$  and therefore it is valid for each  $j$ .

Let us point out in addition that

$$S_j = \sum_{i=0}^j i!(j-i)! = j! \sum_{i=0}^j \frac{1}{\binom{j}{i}},$$

so that from (6) there results the well known formula

$$(9) \quad \sum_{i=0}^j \frac{1}{\binom{j}{i}} = \frac{j+1}{2^{j+1}} \left( 2 + \frac{2^2}{2} + \dots + \frac{2^{j+1}}{j+1} \right).$$

### 3.

To be able to prove the relation (7) let us take into account that through a successive development of the determinant according to the elements of the last column we easily get the relation

$$(10) \quad D_\gamma = (\gamma-1)! + \sum_{j=1}^{\gamma-1} (-1)^{j-1} \binom{\gamma}{j} D_{\gamma-j},$$

so that we may use the mathematical induction. For  $\gamma = 1$  the relation (7) evidently holds. Let us therefore assume that it holds for the indices 1 to  $(\gamma - 1)$  and let us calculate

$$\begin{aligned} D_\gamma &= (\gamma - 1)! + (-1)^{\gamma-1} \sum_{j=1}^{\gamma-1} (-1)^j \binom{\gamma}{j} D_j = (\gamma - 1)! + \\ &+ (-1)^{\gamma-1} \sum_{j=1}^{\gamma-1} (-1)^j \binom{\gamma}{j} \sum_{i=1}^j \binom{j}{i} (i - 1)! = \\ &= (\gamma - 1)! + (-1)^{\gamma-1} \sum_{i=1}^{\gamma-1} (i - 1)! \sum_{j=i}^{\gamma-1} (-1)^j \binom{j}{i} \binom{\gamma}{j}. \end{aligned}$$

Now the well known formula

$$\binom{n + \mu}{n} \binom{x}{n + \mu} = \binom{x}{n} \binom{x - n}{\mu}$$

gives

$$\begin{aligned} D_\gamma &= (\gamma - 1)! + (-1)^{\gamma-1} \sum_{i=1}^{\gamma-1} \binom{\gamma}{i} (i - 1)! \sum_{j=i}^{\gamma-1} (-1)^j \binom{\gamma - i}{j - i} = \\ &= (\gamma - 1)! + (-1)^{\gamma-1} \sum_{i=1}^{\gamma-1} (-1)^i \binom{\gamma}{i} (i - 1)! \sum_{j=0}^{\gamma-i-1} (-1)^j \binom{\gamma - i}{j - i}, \end{aligned}$$

and with the help of another relation

$$\sum_{k=0}^n (-1)^k \binom{x}{k} = \binom{n - x}{n}$$

we further get

$$D_\gamma = (\gamma - 1)! + \sum_{i=1}^{\gamma-1} \binom{\gamma}{i} (i - 1)! (-1)^{\gamma-i-1} \binom{-1}{\gamma - i - 1}.$$

And as

$$(-1)^n \binom{-1}{n} = 1,$$

we finally obtain

$$D_\gamma = (\gamma - 1)! + \sum_{i=1}^{\gamma-1} \binom{\gamma}{i} (i - 1)! = \sum_{j=1}^{\gamma} \binom{\gamma}{j} (j - 1)!,$$

which is the required relation.