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ON CONVERGENCE OF TRANSFINITE SEQUENCES

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In paper [1] relations of some kinds of the convergence of functional sequences are considered. We shall be interested in a similar question, but we shall consider transfinite functional sequences.

Let X be a T_1-topological space which satisfies the first axiom of countability. The sequence $\{x_\xi\}_{\xi<\beta}$, where $x_\xi \in X$ and the index ξ belongs to the set of all ordinal numbers less than the ordinal number β, will be called the tranfinite sequence of the type β (t.s.t. β). If a t.s.t. β ($\beta \geqslant \Omega$, Ω — the first uncountable ordinal) converges to the point x ($\lim\limits_{\xi\to\beta} x_\xi = x$, $x_\xi \to x$), then the point x is unique and there exists μ ($\mu < \beta$) such that for each $\xi \geqslant \mu$ $x_\xi = x$ holds.

Further we shall consider t.s.t. Ω.

Let $X \neq \emptyset$ and (Y, σ) be a metric space which has two elements at least and let $\{f_\xi\}_{\xi<\Omega}$ be a functional t.s.t. Ω ($f_\xi : X \to Y$).

Definition 1. *We shall say that a functional t. s. t. Ω $\{f_\xi\}_{\xi<\Omega}$ converges pointwise (on X) to a function f if for each $x \in X$ $\lim\limits_{\xi\to\Omega} f_\xi(x) = f(x)$.*

Definition of the pointwise convergence (of a transfinite sequence of real functions of the real variable) appeared and the basic properties were studied in paper [3].

Definition 2. *We shall say that a functional t. s. t. Ω $\{f_\xi\}_{\xi<\Omega}$ converges (on X) uniformly to a function f if for each $\varepsilon > 0$ there exists $\mu(\mu < \Omega)$ such that for every $\xi \geqslant \mu$ and each $x \in X$ $\sigma(f_\xi(x), f(x)) < \varepsilon$ holds.*

Definition of uniform convergence (of a transfinite sequence of real functions of the real variable) and some of its properties are considered in paper [2].

Definition 3. *Let X be a topological space. We shall say that a functional t.s.t. Ω $\{f_\xi\}_{\xi<\Omega}$ converges (on X) locally uniformly to f if for each compact C ($C \subset X$) the functional t.s.t. Ω $\{f_\xi \mid C\}_{\xi<\Omega}$ converges uniformly to $f|C$.*

Definition 4. *We shall say that a functional t.s.t. Ω $\{f_\xi\}_{\xi<\Omega}$ converges* *(on X) quasi-uniformly to f if for each $x \in X$ $\lim_{\xi \to \Omega} f_\xi(x) = f(x)$ and to every $\varepsilon > 0$ and every $\eta_0 < \Omega$ there exists η, $\eta_0 < \eta < \Omega$, such that $\inf_{\eta_0 < \xi \leq \eta} \sigma(f_\xi(x), f(x)) < \varepsilon$ for each $x \in X$.*

It is easy to see that the introduced kinds of convergences are introduced analogically to the well-known types of convergences of sequences of the type ω (ω — the first non-finite ordinal). Let (A) and (B) be two of the introduced kinds of convergences. Futher we shall say that the convergences (A) and (B) are equivalent (on X) if a functional t.s.t. Ω $\{f_\xi\}_{\xi<\Omega}$ converges to f (on X) in the sense (A) if and only if it converges to f in the sense (B).

The following theorem gives necessary and sufficient conditions (sufficient conditions, respectively necessary and sufficient conditions for some classes of spaces) for the equivalence of the introduced types of convergences.

Theorem 1. *Let the sets X and Y have the introduced meaning.*

(i) *Pointwise and uniform convergences are equivalent (on X) if and only if X is a countable set.*

(ii) *Let X be a topological space. Pointwise and locally uniform convergences are equivalent (on X) if and only if X has the following property: Every compact C ($C \subset X$) is a countable set.*

(iii) *Pointwise and quasi-uniform convergences are equivalent (on X) if and only if X is a countable set.*

(iv) *Uniform and quasi-uniform convergences are equivalent (on X) if and only if X is a countable set.*

(v) *Let X be a topological space the set of points of condensation X^c of which is void. Then pointwise and locally uniform convergences are equivalent (on X).*

(vi) *Let X be a topological space and let $X = \bigcup_{n<\omega} C_n$ (C_n, $n < \omega$, are compacts). Then uniform and locally uniform convergences are equivalent (on X).*

(vii) *Let X be a countable topological space. Then locally uniform and quasi--uniform convergences are equivalent (on X).*

(viii) *Let X be a locally compact topological space. Then pointwise and locally uniform convergences are equivalent (on X) if and only if $X^c = \emptyset$.*

(ix) *Let X be a topological space with the property: Every compact is a countable set. Then locally uniform and quasi-uniform convergences are equivalent (on X) if and only if X is a countable set.*

For the proof of Theorem 1 we shall use the following lemma.

Lemma. *A functional t.s.t. Ω $\{f_\xi\}_{\xi<\Omega}$ converges (on X) uniformly to f if and only if there exists μ ($\mu < \Omega$) such that for every $\xi \geqslant \mu$ and each $x \in X$ $f_\xi(x) = f(x)$.*

Proof of Theorem 1. (i): If X is a countable set, then the equivalence of pointwise and uniform convergences is obvious (with respect to the Lemma).

Let X be an uncountable set. Then it is possible to choose from X a t.s.t. Ω $\{x_\xi\}_{\xi<\Omega}$ of different terms. Let $a, b \in Y$, $a \neq b$. Let us construct a functional t.s.t. Ω in the following way: $f_\xi(x_\eta) = b$ for $\eta : \xi \leqslant \eta < \Omega$ and $f_\xi(x) = a$ for $x \in X - \bigcup_{\xi \leq \eta < \Omega} \{x_\eta\}$. Obviously for each $x \in X$ $\lim_{\xi \to \Omega} f_\xi(x) = a$, hence the functional t.s.t. Ω $\{f_\xi\}_{\xi<\Omega}$ converges pointwise to $f(x) = a$. It is possible to see with respect to the Lemma that $\{f_\xi\}_{\xi<\Omega}$ does not converge to f uniformly, because for every μ $(\mu < \Omega)$ $f_\mu(x_\mu) = b \neq f(x_\mu)$ holds.

(ii): Let every compact C $(C \subset X)$ be a countable set. Then according to (i) pointwise and uniform convergences on every compact are equivalent, therefore pointwise and locally uniform convergences are equivalent (on X).

If there exists a compact C $(C \subset X)$, which is not a countable set, then with respect to (i) there exists on C a convergent functional t.s.t. Ω $\{g_\xi\}_{\xi<\Omega}$ and a function g such that the convergence $g_\xi \to g$ is pointwise but not uniform. Then the functional t.s.t. Ω $\{f_\xi\}_{\xi<\Omega}$ of functions on X $(f_\xi | C = g_\xi$, $f_\xi(x) = a$ for $x \notin C)$ converges pointwise to $f (f | C = g, f(x) = a$ for $x \notin C)$ but not locally uniform. Hence pointwise and locally uniform convergences are not equivalent (on X).

(iii): We shall use the following property of quasi-uniform convergence: A functional t.s.t. Ω $\{f_\xi\}_{\xi<\Omega}$ converges to f quasi-uniformly if and only if for each $x \in X$ $\lim_{\xi \to \Omega} f_\xi(x) = f(x)$ and for every $\eta_0 < \Omega$ there exists η, $\eta_0 < \eta < \Omega$ such that $\inf_{\eta_0 < \xi < \eta} \sigma(f_\xi(x), f(x)) = 0$ holds for each $x \in X$.

Let X be a countable set. It suffices to show that from the pointwise convergence of a functional t.s.t. Ω $\{f_\xi\}_{\xi<\Omega}$ there follows the quasi-uniform convergence. We can easy verify that there is μ $(\mu < \Omega)$ such that for every $\xi \geqslant \mu$ and each $x \in X$ $f_\xi(x) = f(x)$ holds. Therefore $\lim_{\xi \to \Omega} f_\xi(x) = f(x)$ holds for each $x \in X$ and also for any $\eta_0 < \Omega$ $\inf_{\eta_0 < \xi < \eta} \sigma(f_\xi(x), f(x)) = 0$, where $\eta = \max\{\mu, \eta_0 + 1\}$, hence $f_\xi \to f$ quasi-uniformly.

Let X be an uncountable set. Then we can repeat the construction of the functional t.s.t. Ω $\{f_\xi\}_{\xi<\Omega}$ from part (i). This functional t.s.t. Ω converges pointwise, but it does not converge quasi-uniformly, because there exists η_0 $(\eta_0 = 1)$ such that for every $\eta < \Omega$ $\inf_{\eta_0 < \xi \leq \eta} \sigma(f_\xi(x_\eta), f(x_\eta)) = \sigma(a, b) > 0$ holds.

(iv): If X is a countable set then the equivalence of uniform and quasi-uniform convergences is evident.

Let X be an uncountable set. Then it is possible to choose in X a t.s.t. Ω $\{x_\xi\}_{\xi<\Omega}$ of different terms. Let $a, b \in Y$, $a \neq b$. Let us construct a functional t.s.t. Ω $\{f_\xi\}_{\xi<\Omega}$ as follows: $f_\xi(x_\xi) = b$ and $f_\xi(x) = a$ for $x \neq x_\xi$. We show that $f_\xi \to f$ $(f(x) = a)$ quasi-uniformly. Indeed, for each $x \in X$ $\lim_{\xi \to \Omega} f_\xi(x) = f(x)$ and for any $\eta_0 < \Omega$ there exists $\eta = \eta_0 + 2$ such that $\inf\limits_{\eta_0 < \xi \leq \eta} \sigma(f_\xi(x), f(x)) = 0$ holds for each $x \in X$. But from the Lemma it follows immediately that the convergence $f_\xi \to f$ is not uniform.

(v): With respect to (ii) it is sufficient to prove the statement: If X is a topological space with the property $X^c = \emptyset$, then every compact C ($C \subset X$) is a countable set. The assumption $X^c = \emptyset$ implies that for each $x \in X$ there exists its open neighbourhood U_x such that U_x is a countable set. If A ($A \subset X$) is any uncountable set then the family $\{U_x : x \in A\}$ is an open cover of A from which is not possible to choose a finite subcover. Hence A is not a compact.

(vi): The proof follows immediately from the Lemma.

(vii): Let a functional t.s.t. Ω $\{f_\xi\}_{\xi < \Omega}$ converge on X locally uniformly to f. Then (because a singleton $\{x\}$ is compact) $\lim\limits_{\xi \to \Omega} f_\xi(x) = f(x)$ holds for each $x \in X$. It is easy to see that there exists $\mu < \Omega$ such that for any $\xi \geqslant \mu$ $f_\xi(x) = f(x)$ holds for each $x \in X$. Therefore according to Lemma $f_\xi \to f$ uniformly and with respect to (iv) $f_\xi \to f$ quasi-uniformly. If $f_\xi \to f$ quasi-uniformly, then according to (iii) $f_\xi \to f$ pointwise and with respect to (ii) $f_\xi \to f$ locally uniformly.

(viii): The proof of (viii) is a consequence of (ii) and of the following statement: If X is a locally compact topological space then every compact C ($C \subset X$) is countable if and only if $X^c = \emptyset$. Indeed, if $X^c = \emptyset$, we can show in the same way as in the proof of (v) that each compact C ($C \subset X$) is countable. If $X^c \neq \emptyset$, then for each $x \in X^c$ there exists its neighbourhood U_x such that U_x is an uncountable compact.

(ix): If X is a countable set, then according to (vii) locally uniform and quasi-uniform convergences are equivalent.

If X is an uncountable set, then it is possible to choose in X a t.s.t. Ω of different terms. Let $a, b \in Y$, $a \neq b$. We construct the functional t.s.t. Ω $\{f_\xi\}_{\xi < \Omega}$ and f in the same way as in the proof of (i). If C is a compact (according to the assumption the set C is countable) and μ_c the first ordinal with the property $\mu_c > \eta$ for each η in the set $\{\eta : x_\eta \in C\}$, then for every $\xi \geqslant \mu_c$ $f_\xi(x) = f(x)$ holds for each $x \in C$, hence $f_\xi \to f$ locally uniformly. On the other hand there exists $\eta_0 < \Omega$ ($\eta_0 = 1$) such that for any $\eta < \Omega$ $\inf\limits_{\eta_0 < \xi \leq \eta} \sigma(f_\xi(x_\eta)), f(x_\eta)) = \sigma(a, b) > 0$, therefore the convergence $f_\xi \to f$ is not quasi-uniform.

Theorem 1 is therefore completely proved.

It is easy to see that the condition $X^c = \emptyset$ from part (v) of Theorem 1 is only sufficient but is not necessary to the statement. The following example shows it.

Example 1. Let R be the set of all rational numbers of the interval $\langle 0, 1\rangle$ and let $\{X_r : r \in R\}$ be a family of sets with these properties: X_0 is one-point set ($X_0 = \{x_0\}$), for $r > 0$ every set X_r is uncountable and for $r \neq s$ $X_r \cap$

$\cap X_s = \emptyset$. Let us define a metric ϱ on $X = \cup \{X_r : r \in R\}$ in the following way: If $x \in X_r$, $y \in X_s$, $x \neq y$ then $\varrho(x, y) = \max\{r, s\}$, $\varrho(x, x) = 0$. The set $\{x : \varrho(x, x_0) < \varepsilon\} = \cup \{X_r : r \in R, r < \varepsilon\}$ is uncountable for each $\varepsilon > 0$, hence $X^c \neq \emptyset$. Every point $x \neq x_0$ is an isolated point of the space X ($\{y : : \varrho(y, x) < r\} = \{x\}$ for $x \in X_r$). Therefore for any compact C ($C \subset X$) and for each $r \in R$ the set $C \cap X_r$ is finite, hence C is a countable set and with respect to part (ii) of Theorem 1 pointwise and locally uniform convergences on X are equivalent.

Remark 1. Example 1 shows, that the assumption of a locally compactness of the space X in part (viii) of Theorem 1 is essential. Each neighbourhood of x_0 contains an uncountable set X_r ($r \in (0, 1\rangle$). If $x, y \in X_r$, $x \neq y$, then $\varrho(x, y) = r > 0$, hence X is not a locally compact space, $X^c \neq \emptyset$, but pointwise and locally uniform convergences on X are equivalent.

The assumption $X = \bigcup\limits_{n<\omega} C_n$, C_n are compacts, of part (vi) of Theorem 1 is essential. The following example shows it.

Example 2. Let $X = \{\xi : \xi < \Omega\}$, $\varrho(\xi, \eta) = 1$ for $\xi \neq \eta$ and $\varrho(\xi, \xi) = 0$. Let $a, b \in Y$, $a \neq b$. Let us define the functional t.s.t. Ω $\{f_\xi\}_{\xi<\Omega}$ as follows: $f_\xi(\xi) = b$, $f_\xi(\eta) = a$ for $\eta \neq \xi$. This functional t.s.t. Ω converges locally uniformly to the function $f(x) = a$. Indeed, C is a compact in (X, ϱ) if and only if C is a finite set. If $\mu_c = \max\{x : x \in C\}$ then for $\xi > \mu_c$ $f_\xi(x) = f(x)$ holds for $x \in C$. From the Lemma there follows the locally uniform convergence $f_\xi \to f$. But from the Lemma it follows immediately that the convergence $f_\xi \to f$ on X is not uniform.

Remark 2. The assumption of the countability of the set X in part (vii) of Theorem 1 is essential. It follows immediately from the proof of part (ix).

Remark 3. Sequences of the type ω are considered also with respect to the continuous convergence. It would be possible to define this type of convergence for a functional t.s.t. Ω as follows: Let X and Y be metric spaces and let $\{f_\xi\}_{\xi<\Omega}$ ($f_\xi : X \to Y$) be functional t.s.t. Ω. We shall say that $f_\xi \to f$ ($f : X \to Y$) continuously, if for each convergent t.s.t. Ω $\{x_\xi\}_{\xi<\Omega}$, $x_\xi \to x$, $f_\xi(x_\xi) \to f(x)$ holds. However, it is easy to see that continuous and pointwise convergences are equivalent on any X. It suffices to show that pointwise convergence implies continuous convergence. If $x_\xi \to x$, then there exists $\mu < \Omega$ such that $x_\xi = x$ holds for every $\xi \geqslant \mu$, therefore for $\xi \geqslant \mu$ $f_\xi(x_\xi) = f_\xi(x) \to f(x)$.

In Theorem 1 the question of equivalence of functional t.s.t. Ω is not solved completely. It woud be interesting to solve this question: Is the sufficient condition of part (vi) (of part (vii)) of Theorem 1 also the necessary condition for the equivalence of convergences? If not, then find the necessary and sufficient condition.

Futher we shall consider a metric space Z of transfinite sequence of the type Ω.

Theorem 2. *Let* (X, σ) *be a metric space and* $Z = \bigtimes_{\xi<\Omega} X_\xi$ $(X_\xi = X,\ \xi < \Omega)$ *be a metric space with the metric* $\varrho(a, b) = \sup_\xi \{\min\}\{1, \sigma(a_\xi, b_\xi)\}\}$ $(a = \{a_\xi\}_{\xi<\Omega}$, $b = \{b_\xi\}_{\xi<\Omega})$. *Let* $A_x = \{a = \{a_\xi\}_{\xi<\Omega} \in Z : \lim_{\xi\to\Omega} a_\xi = x\}$ $(x \in X)$. *Then the set* A_x *is closed for each* $x \in X$.

Proof. Let $x \in X$ and let $a^m = \{a_\xi^m\}_{\xi<\Omega} \to a = \{a_\xi\}_{\xi<\Omega} (a^m \in A_x)$. We show that $a \in A_x$. For every $m = 1, 2, \ldots$ there exists the ordinal α_m $(\alpha_m < \Omega)$ such that $a_\xi^m = x$ holds whenever $\alpha_m \leqslant \xi < \Omega$. Let α be the first ordinal, greater than each α_m $(m = 1, 2, \ldots)$. Then $\alpha < \Omega$ and for every ξ, $\alpha \leqslant \xi < \Omega$ the sequence $\{a_\xi^m\}_{m=1}^\infty$ is constant $(a_\xi^m = x)$. Obviously for each ξ, $\alpha \leqslant \xi < \Omega$, $a_\xi = x$. If there exists ξ_0 $(\alpha \leqslant \xi_0 < \Omega)$ such that $a_{\xi_0} \neq x$, then $\lim_{m\to\infty} a_{\xi_0}^m = x \neq a_{\xi_0}$ and it is a contradiction to the assumption $a^m \to a$. Hence $\lim_{\xi\to\Omega} a_\xi = x$, $a \in A_x$.

Remark 4. The statement of Theorem 2 does not hold if the set $Z = \bigtimes_{\xi<\Omega} X_\xi$ $(X_\xi = X$, X has two points at least) is considered with respect to the product topology. It is easy to show that for any points $x, y \in X$ $\overline{A}_x \cap A_y \neq \emptyset$ holds ($\overline{M}$ — closure of the set M). We construct a net $\{a^\gamma, \gamma \in I\}$ $(I = \{\gamma : \gamma < \Omega\})$, $a^\gamma = \{a_\xi^\gamma\}_{\xi<\Omega} \in A_x$, $a^\gamma \to a = \{a_\xi\}_{\xi<\Omega} \in A_y$ as follows: Let us put $a_\xi^\gamma = y$ for $\xi < \gamma$ and $a_\xi^\gamma = x$ for $\gamma \leqslant \xi$. Obviously $\lim_{\xi\to\Omega} a_\xi^\gamma = x$, hence $a^\gamma \in A_x$ for each $\gamma \in I$. Since for each $\xi \in I$ $\lim_{\gamma\to\Omega} a_\xi^\gamma = y$ and the convergence in the product topology is the pointwise convergence $a = \{y\}_{\xi<\Omega} \in A_y$.

Theorem 3. *Let* (X, σ) *be a metric space without isolated points and let the space* (Z, ϱ) *and the sets* A_x $(x \in X)$ *have the same meaning like in Theorem 2. Then the set* $A = \bigcup_{x\in X} A_x$ *has the complement dense in* Z *and every of the sets* A_x *is nowhere dense in* Z.

Proof. Let $\varepsilon > 0$. We show that every open sphere $K(a, \varepsilon) = \{b : \varrho(a, b) < \varepsilon\} \subset Z$ contains an element of $Z - A$. We can suppose $a = \{a_\xi\}_{\xi<\Omega} \in A$. Hence there is $x \in X$ such that $\lim_{\xi\to\Omega} a_\xi = x$. As X has no isolated points (according to the assumption) there exists $y \in X$, $y \neq x$ such that $\sigma(x, y) < \varepsilon$. Let α be the first ordinal with the property: For each ξ, $\alpha \leqslant \xi < \Omega$ $a_\xi = x$ holds. Let I_1 and I_2 be disjoint cofinal subsets of the set $I = \{\xi : \xi < \Omega\}$. Let $I_1 \cup I_2 = I$ and let for each $\xi \in I_2$ $\alpha \leqslant \xi$ hold. Then the element $b = \{b_\xi\}_{\xi<\Omega}$ $(b_\xi = a_\xi$ for $\xi \in I_1$ and $b_\xi = y$ for $\xi \in I_2)$ belongs so $Z - A$ and $\varrho(a, b) \leqslant \sigma(x, y) < \varepsilon$.

To prove the second part of the statement of Theorem 3 it suffices to show that for each $x \in X$ the set $Z - \bar{A}_x$ is dense in Z. Since $A_x = \bar{A}_x$ (according to Theorem 2) the ralations $Z - \bar{A}_x = Z - A_x \supset Z - A$ hold. The last set is according to the first part of the statement dense in Z, therefore $Z - \bar{A}_x$ is also dense in Z.

Theorem 4. *Let X be a countable set and let (X, σ) be a metric space without isolated points. Let the space (Z, ϱ) and the set A have the same meaning like in Theorem 3. Then A is F_σ and of the first category in Z.*

Proof. According to the assumption $X = \bigcup_{n=1}^{\infty} \{x_n\}$, hence $A = \bigcup_{n=1}^{\infty} A_{x_n}$. With respect so Theorem 2 each of the sets A_{x_n} $(n = 1, 2, \ldots)$ is closed — A is F_σ in Z. From Theorem 3 it follows that each of the sets A_{x_n} $(n = 1, 2, \ldots)$ is nowhere dense — A is of the first category in Z.

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SÚŤAŽ MLADÝCH PRACOVNÍKOV V MATEMATIKE

Jednota slovenských matematikov a fyzikov vypisuje na rok 1972 súťaž mladých pracovníkov v matematike.

Súťaže sa môžu zúčastniť členovia JSMF, ktorých vek v roku 1972 neprekročí 30 rokov. Do súťaže sa prijímajú vedecké práce z matematiky (jednotlivé alebo súbor prác), ktoré boli publikované alebo prijaté redakčnou radou niektorého odborného časopisu.

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