Bohdan Zelinka

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THE DECOMPOSITION OF A DIGRAPH INTO ISOTOPIC SUBGRAPHS

BOHDAN ZELINKA, Liberec

Analogously to the concept of isotopy of groupoids defined in [1] we shall define the concept of isotopy of digraphs.

Definition. *Two digraphs G_1 and G_2 are isotopic to each other if and only if there exist two one-to-one mappings f_1, f_2 of the vertex set V_1 of G_1 onto the vertex set V_2 of G_2 such that for any two vertices u, v of G_1 the existence of the edge $\overrightarrow{f_1(u)f_2(v)}$ in G_2 is equivalent to the existence of the edge $\overrightarrow{uv}$ in G_1. The ordered pair of mappings $\mathfrak{f} = \langle f_1, f_2 \rangle$ is called an isotopy of G_1 onto G_2. An isotopy of a digraph G onto itself is called an autotopy of G.* If $f_1 \equiv f_2$, the isotopy $\mathfrak{f} = \langle f_1, f_2 \rangle$ is an isomorphism of G_1 onto G_2. Here analogously to [2] we shall study decompositions of a digraph into two isotopic edge-disjoint subgraphs. In the first paragraph we shall investigate complete digraphs, in the second digraphs which are not necessarily complete.

§ 1.

In this paragraph we shall study decompositions of complete digraphs into two isotopic edge-disjoint subgraphs. We shall distinguish a complete digraph $\overrightarrow{K_n}$ (with n vertices) without loops and a complete digraph $\overrightarrow{K_n^0}$ with loops. The number n is an arbitrary cardinal number. In $\overrightarrow{K_n^0}$ for each ordered pair of vertices $[u, v]$ the edge $\overrightarrow{uv}$ exists. In $\overrightarrow{K_n}$ there exists the edge $\overrightarrow{uv}$ only for each ordered pair of different vertices $[u, v]$, $u \neq v$. For the investigation of decompositions of digraphs into *isomorphic* subgraphs this difference is only slight, thus in [2] the digraphs $\overrightarrow{K_n^0}$ were not studied. But when we study decompositions of a digraph into *isotopic* subgraphs, this difference is essential. We shall prove a theorem concerning $\overrightarrow{K_n}$.

Theorem 1. *Any autotopy of a complete digraph $\overrightarrow{K_n}$ with n vertices without loops is an automorphism of $\overrightarrow{K_n}$.*

Proof. Let $\mathfrak{f} = \langle f_1, f_2 \rangle$ be an autotopy of $\overrightarrow{K_n}$. If $\mathfrak{f}$ is not an automorphism of $\overrightarrow{K_n}$, then $f_1 \not\equiv f_2$, i. e. there exists a vertex u of $\overrightarrow{K_n}$ such that $f_1(u) \neq f_2(u)$.

As the vertices $f_1(u)$, $f_2(u)$ are different, there exists the edge $\overrightarrow{f_1(u)f_2(u)}$ in $\overrightarrow{K_n}$. According to the definition of isotopy also the edge $\overrightarrow{uu}$ must exist in $\overrightarrow{K_n}$, but this is a loop and $\overrightarrow{K_n}$ is a complete digraph without loops. We have obtained a contradiction.

Thus all the following considerations will concern only complete digraphs with loops.

We say that a complete digraph $\overrightarrow{K_n^0}$ with n vertices with loops is decomposed into two isotopic edge-disjoint subgraphs G and $\bar{G}$, if and only if there exist two edge-disjoint subgraphs G and $\bar{G}$ of $\overrightarrow{K_n^0}$ such that they are isotopic to each other, each of them contains all vertices of $\overrightarrow{K_n^0}$ and their union is $\overrightarrow{K_n^0}$.

Theorem 2. *Let $\overrightarrow{K_n^0}$ be decomposed into two isotopic edge-disjoint subgraphs G and $\bar{G}$, let $\mathfrak{f} = \langle f_1, f_2 \rangle$ be an istotopy of G onto $\bar{G}$. Then at least one of the permutations f_1, f_2 has only cycles of even or infinite lengths.*

Remark. An isotopy of G onto $\bar{G}$ is evidently an autotopy of $\overrightarrow{K_n^0}$. The mappings f_1, f_2 are evidently permutations of the vertex set V of $\overrightarrow{K_n^0}$.

Proof. Assume that both the permutations f_1, f_2 have cycles of odd lengths. Let $\mathscr{C}$ be such a cycle of f_1, $\mathscr{C}'$ such a cycle of f_2, Let c, c' consequently be the lengths of $\mathscr{C}$ and $\mathscr{C}'$ they are odd numbers. Let u be a vertex of $\mathscr{C}$, v a vertex of $\mathscr{C}'$. Assume without the loss of generality that the edge $\overrightarrow{uv}$ belongs to G. Then evidently the edge $\overrightarrow{f_1(u)f_2(v)}$ belongs to $\bar{G}$, the edge $\overrightarrow{f_1^2(u)f_2^2(v)}$ belongs to G, generally, for k odd the edge $\overrightarrow{f_1^k(u)f_2^k(v)}$ belongs to $\bar{G}$, for an even k it belongs to G. Let r be the least common multiple of c and c', it is also an odd number. Then $\overrightarrow{f_1^r(u)f_2^r(v)}$ belongs to $\bar{G}$. But as r is a multiple of the length c of the cycle $\mathscr{C}$ of the permutation f_1, we have $f_1^r(u) = u$ and analogously, as r is also a multiple of c', we have $f_2^r(v) = v$. Thus $\overrightarrow{f_1^r(u)f_2^r(v)} = \overrightarrow{uv}$ and it belongs to G. Thus $\overrightarrow{uv}$ belongs to both G and $\bar{G}$, which is a contradiction with the assumption that G and $\bar{G}$ are edge-disjoint.

Corollary 1. *A digraph $\overrightarrow{K_n^0}$ where n is odd, cannot be decomposed into two isotopic edge-disjoint subgraphs.*

Evidently, if some set has an odd number of elements, each permutation of it must have at least one cycle of an odd length.

This corollary can be proved also in another way. The digraph $\overrightarrow{K_n^0}$ has n^2 edges. As G and $\bar{G}$ are isotopic, they must have equal numbers of edges; as they are edge-disjoint and their union is $\overrightarrow{K_n^0}$, the sum of their numbers of edges must be n^2. Thus each of them contains exactly $\frac{1}{2}n^2$ edges. If n is odd, also n^2 is odd and $\frac{1}{2}n^2$ is not an integer., thus G and $\bar{G}$ cannot exist.

Theorem 3. *Let V be the vertex set of $\overrightarrow{K_n^0}$, let f_1 be a permutation of V having only cycles of even and infinite lengths, let f_2 be an arbitrary permutation of V. Then there exists a decomposition of $\overrightarrow{K_n^0}$ into two istotopic edge-disjoint subgraphs G and $\bar{G}$ such that $\mathfrak{f} = \langle f_1, f_2 \rangle$ is an isotopy of G onto $\bar{G}$.*

Proof. We shall construct the graphs G and $\bar{G}$. Choose an arbitrary edge $\overrightarrow{u_1v_1}$ in $\overrightarrow{K_n^0}$ and put it into G. Then for any even k put the edge $\overrightarrow{f_1^k(u_1)f_2^k(v_1)}$ into G and for any odd k put that edge into $\bar{G}$. We shall prove by a contradiction that no edge will be put at the same time into both G and $\bar{G}$. Assume that this case occurs, i. e. that there exists an even integer k and an odd integer l so that $\overrightarrow{f_1^k(u_1)f_2^k(v_1)} = \overrightarrow{f_1^l(u_1)f_2^l(v_1)}$. Thus $f_1^k(u_1) = f_1^l(u_1)$, $f_2^k(v_1) = f_2^l(v_1)$. We have $f_1^l \equiv f_1^k f_1^{l-k}$, therefore $f_1^k(u_1) = f_1^l(u_1)$ (or $f_2^k(v_1) = f_2^l(v_1)$, respectively) implies that the length of the cycle of f_1 (or of f_2, respectively) containing u_1 (or v_1, respectively) is a divisor of the nubmer $k - l$. The number $k - l$ is a difference of an even number and an odd one, thus it is odd and all its divisors are odd. Therefore the cycle of f_1 containing u_1 and the cycle of f_2 containing v_1 both have odd lengths. which is a contradiction. Then choose an edge $\overrightarrow{u_2v_2}$, which has not been put yet into any of the digraphs G and $\bar{G}$ and repeat the procedure with it instead of $\overrightarrow{u_1v_1}$. We shall prove by a contradiction that for any k the edge $\overrightarrow{f_1^k(u_2)f_2^k(v_2)}$ is not identical with $\overrightarrow{f_1^l(u_1)f_2^l(v_1)}$ for any l. Assume that $\overrightarrow{f_1^k(u_2)f_2^k(v_2)} = \overrightarrow{f_1^l(u_1)f_2^l(u_1)}$ for scme k and l. Then $\overrightarrow{f_1^{l-k}(u_1)f_2^{l-k}(v_1)} = \overrightarrow{u_2v_2}$ and $\overrightarrow{u_2v_2}$ is put into G, if $l - k$ is even, or into $\bar{G}$, if $l - k$ is odd. This is a contradiction with the asumption that $\overrightarrow{u_2v_2}$ has not been put yet into any of the digraphs G and $\bar{G}$. Then we choose again and edge $\overrightarrow{u_3v_3}$ which has not been put yet into any of the digraphs G and $\bar{G}$ and continue the procedure with it, then we choose $\overrightarrow{u_4v_4}$ etc. We do this procedure until each edge of $\overrightarrow{K_n^0}$ is put either into G, or into $\bar{G}$.

Remark. We use also negative exponents at f_1 and f_2. By f_1^{-k}, where k is positive, we denote the inverse mapping to f_1^k, by f_1^0 we denote the identical mapping. Analogously for f_2.

Corollary 2. *Any digraph $\overrightarrow{K_n^0}$ where n is even or infinite, can be decomposed into two isotopic edge-disjoint subgraphs.*

Theorem 3'. *Let V be the vertex set of $\overrightarrow{K_n^0}$. Let f_2 be a permutation of V having only cycles of even of infinite lengths, let f_1 be an arbitrary permutation of V. Then there exists a decomposition of $\overrightarrow{K_n^0}$ into two isotopic edge-disjoint subfraphs G and $\bar{G}$ such that $\mathfrak{f} = \langle f_1, f_2 \rangle$ is an isotopy of G onto $\bar{G}$.*

Proof is dual to the proof of Theorem 3.

§ 2.

In this paragraph we shall describe a decomposition of a finite digraph G (which is not necessarily a complete digraph) into two isotopic edge-disjoint subgraphs, each of which contains all vertices of the digraph G and whose union is G. We shall consider only such decompositions, where an isotopy of one graph onto the other is an autotopy of G. A digraph which can be decomposed in this way will be called an R_2'-digraph (analogously to the concept of a R_2-digraph from [2]).

The vertex set of the graph G will be denoted by U, its edge set by H. The isotopic subgraphs which the digraph G is decomposed into will be denoted by G_1, G_2, their edge sets consequently by H_1, H_2. The complement of G (with respect to the complete digraph with loops) will be denoted by $\bar{G}$, its edge set by $\bar{H}$. The isotopy of G_1 onto G_2 will be denoted by $\mathfrak{f} = \langle f_1, f_2 \rangle$. First we shall express some lemmas about $\mathfrak{f}$.

Lemma 1. *If a cycle $\mathscr{C}_1$ of f_1 and a cycle $\mathscr{C}_2$ of f_2 have both odd lengths, then there exists no edge in G whose initial vertex would belong to $\mathscr{C}_1$ and whose terminal vertex would belong to $\mathscr{C}_2$.*

Proof. Let there exist an edge from a vertex of $\mathscr{C}_1$ into a vertex of $\mathscr{C}_2$; denote it by $\overrightarrow{u_1v_1}$. As $H_1 \cup H_2 = H$, $H_1 \cap H_2 = \emptyset$, the edge $\overrightarrow{uv}$ belongs either to H_1, or to H_2. Without the loss of generality assume that it belongs to H_1. We have now $\overrightarrow{f_1^k(u)f_2^k(v)} \in H_1$ for k even and $\overrightarrow{f_1^k(u)f_2^k(v)} \in H_2$ for k odd. Let the length of $\mathscr{C}_1$ be c_1 and the length of $\mathscr{C}_2$ be c_2; they are both odd numbers. Their least common multiple g will be also an odd number. Therefore $\overrightarrow{f_1^q(u)f_2^q(v)} \in H_2$. But, as q is a multiple of c_1, we have $f_1^q(u) = u$ and as q is a multiple of c_2, we have $f_2^q(v) = v$. Thus $\overrightarrow{f_1^q(u)f_2^q(v)} = \overrightarrow{uv}$ and it belongs to both H_1 and H_2, which is impossible, because $H_1 \cap H_2 = \emptyset$.

The autotopy $\mathfrak{f} = \langle f_1, f_2 \rangle$ of the digraph G can be considered as a mapping of $U \times U$ onto itself, assigning to each ordered pair $[u, v] \in U \times U$ the ordered pair $[f_1(u), f_2(v)]$.

Lemma 2. *Let $\hat{\mathscr{C}}$ be a cycle of* $\mathfrak{f}$. *If $\hat{\mathscr{C}}$ contains a pair of vertices $[u, v]$ of U such that either u does not belong to a cycle of f_1 of an odd length, or v does not belong to a cycle of f_2 of an odd length, then the cycle $\hat{\mathscr{C}}$ has an even length.*

Proof. Let g be the lenght of $\hat{\mathscr{C}}$, let c_1 (or c_2, respectively) be the length of the cycle of f_1 (or f_2, respectively) containing u (or v, respectively). Then $f_1^q(u) = u$, $f_2^q(v) = v$ and g is a common multiple of c_1 and c_2. As at least one of the numbers c_1, c_2 is even, g is also even.

Now let us have a vertex set U and the permutations f_1, f_2 on it. We shall describe a construction which will be called a construction $\mathscr{C}$. If a vertex u belongs to a cycle of f_1 of an odd length and a vertex v belongs to a cycle of f_2 of an odd length, the edge $\overrightarrow{uv}$ will be in $\bar{H}$. Then we choose an ordered pair of vertices $[u_1, v_1]$ such that the edge $\overrightarrow{u_1v_1}$ has not been put yet into $\bar{H}$ and we put the edge $\overrightarrow{u_1v_1}$ into an arbitrary one of the sets $H_1, H_2, \bar{H}$. If $\overrightarrow{u_1v_1} \in H_1$ (or $\overrightarrow{u_1v_1} \in H_2$, respectively), we put the edge $\overrightarrow{f_1^k(u_1)f_2^k(v_1)}$ into H_1 (or into H_2, respectively) for k even and into H_2 (or H_1, respectively) for k odd. For $\overrightarrow{u_1v_1} \in \bar{H}$ we put $\overrightarrow{f_1^k(u_1)f_2^k(v_1)}$ into $\bar{H}$ for each k. Then we choose another ordered pair

$[u_2, v_2]$ of vertices such that on $\overrightarrow{u_2v_2}$ it has not been determined yet which set it belongs to and we proceed analogously. In this way we proceed until on each edge it is determined which of the sets $H_1, H_2, \bar{H}$ it belongs to.

An immediate consequence of Lemma 2 is the following theorem.

Theorem 4. *By the construction $\mathscr{C}$ always an R_2'-digraph with the vertex set U is created such that $\mathfrak{f} = \langle f_1, f_2 \rangle$ is the corresponding isotopy.*

Evidently also each of such R_2'-digraphs can be obtained by this construction.

Theorem 5. *If two even positive integers m, n are given, $m \leqslant n^2$, there exists an R_2'-digraph with n vertices and m edges.*

Proof is analogous to the proof of Theorem of [2]. It is in fact a weakening of that theorem (which speaks about isomorphic subgraphs) changed for the case of digraphs with loops.

Now we shall study R_2'-digraphs isotopic with their complements. As we consider only finite digraphs, the number of vertices of such a digraph must be even.

Theorem 6. *Let a finite vertex set V with an even number n of elements and on it two permutations f_1, f_2 be given such that the permutation f_1 has only cycles of even lengths. Then there exists an R_2'-digraph G istootopic with its complement (with respect to a complete digraph with loops), whose vertex set is V such that $\mathfrak{f} = \langle f_1, f_2 \rangle$ is an isotopy of G onto its complement $\bar{G}$.*

Proof. We shall construct such an R_2'-digraph. Let $\mathscr{C}_1, \ldots, \mathscr{C}_p$ be the cycles of the permutation f_1: let the vertices of the cycle $\mathscr{C}_i$ $(i = 1, \ldots, p)$ be $u_1^{(i)}, \ldots, u_{r_i}^{(i)}$ such that $f_1(u_j^{(i)}) = u_{j+1}^{(i)}$ for $j = 1, \ldots, r_i - 1$, $f_1(u_{r_i}^{(i)}) = u_1^{(i)}$. All r_i are even. We lead edges from all $u_j^{(i)}$ for j odd into all vertices of V and no other edges. The resulting digraph will be denoted by G. The pair $\mathfrak{f} = \langle f_1, f_2 \rangle$ is an isotopy of G onto its complement $\bar{G}$, because if e is an edge of G, its initial vertex is $u_j^{(i)}$ for some odd j and some i, its image $f_1(u_j^{(i)})$ is $u_{j+1}^{(i)}$ (the subscript $j + 1$ is taken modulo r_i), thus it is an initial vertex of no edge of G, therefore e is transformed by the isotopy $\mathfrak{f}$ into an edge of $\bar{G}$. Now denote the vertices of V by $v_1, \ldots, v_n$ independently on the preceding notation. Let G_1 (or G_2, respectively) be the subgraph of G consisting of all vertices of G and all edges of G whose terminal vertices have an odd (or even, respecively) subscript at v. The digraphs G_1 and G_2 are evidently edge-disjoint and their union is G. Now define an autotopy $\mathfrak{g} = \langle g_1, g_2 \rangle$ of G. We define g_1 as an identical mapping and $g_2(v_i) = v_{i+1}$ for $i = 1, \ldots, n - 1$, $g_2(v_n) = v_1$. Evidently $\mathfrak{g}$ is an isosopy of G_1 onto G_2.

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