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USING THE COMPUTER TO INVESTIGATE CYCLIC STEINER QUADRUPLE SYSTEMS

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A Steiner quadruple system (briefly SQS) of order n on elements

$$(\mathrm{I}) \qquad a_1, a_2, \ldots, a_n$$

is a system S of quadruples formed from the elements (I) so that each triple which may be formed from the elements (1) is contained in exactly one of the quadruples of the system S.

The number of quadruples in a SQS of order n is $n(n - 1)(n - 2)/24$. It is easy to see that the necessary condition for the existence of a SQS of order n is $n \equiv 2$ or $4 \pmod 6$. It was shown by Hanani in [1] that this condition is also sufficient.

A Steiner quadruple system S of order n is said to be cyclic if it satisfies the condition;

$$(\mathrm{II}) \qquad \text{If } (a_i, a_j, a_k, a_l) \in S, \text{ then } (a_{i+1}, a_{j+1}, a_{k+1}, a_{l+1}) \in S,$$

where the indices are taken modulo n.

The purpose of this article is to present a survey of cyclic SQS up to the order 16 included. The necessary and sufficient condition for the existence of a cyclic SQS of order n remains so far unknown to us.

In the following we write for reasons of brevity only i instead of a_i. We prove the following statements (in the proof of the statements C and D the computer has been used);

A. There exists no cyclic SQS of order 8.

B. There exists exactly one cyclic SQS of order 10.

C. There exists no cyclic SQS of order 14.

D. There exists no cyclic SQS of order 16.

For the sake of completeness let us remark that in the trivial case $n = 4$ the system (1, 2, 3, 4) satisfies the condition (II), consequently it is cyclic.

Let a cyclic SQS of order n on the elements $1, 2, \ldots, n$ be given. To an arbitrary quadruple (i_1, i_2, i_3, i_4), $i_1 \leqq i_2 \leqq i_3 \leqq i_4$, of the system we may uniquely assign an associated quadruple (shortly a-quadruple) $Q = [j_1, j_2, j_3, j_4]$, where $j_k = \min(|i_{k+1} - i_k|, n - |i_{k+1} - i_k|)$, $k = 1, 2, 3, 4$, $i_5 = i_1$. The set of all different quadruples (two quadruples are regarded as different if they differ in at least one element) with the same a-quadruple Q is said to be a cyclic set with the a-quadruple Q and denoted by $M(Q)$; an arbitrary quadruple from $M(Q)$ is the generator of this set in the sense of the condition (II); the number of elements of the cyclic set $M(Q)$ is said to be the period of the a-quadruple Q and denoted by $\pi(Q)$ (cf. [2]; evidently $\pi(Q)$ must be a divisor of n). Two a-quadruples are regarded as equal if they differ only by a cyclic permutation of its elements; in the opposite case a-quadruples are different.

From the condition (II) it follows;

(III) A cyclic SQS either contains simultaneously all the $\pi(Q)$ quadruples or contains no quadruple from the cyclic set $M(Q)$.

Let us turn now to the graph-theoretical interpretation. By a graph we understand an undirected graph without loops (multiple edges are allowed). Denote by the symbol $\langle n, k\rangle$ the graph with n vertices in which two arbirary different vertices are joined by exactly k edges. It is easy to verify that the problem of constructing a SQS of order n (a cyclic SQS of order n) is equivalent to the problem of finding a decomposition (a cyclic decomposition) $K = \{K_1, \ldots, K_r\}$ of the graph $\left\langle n, \dfrac{n-2}{2}\right\rangle$ into 4-vertex-cliques (1) so that each triangle with the vertices from $\left\langle n, \dfrac{n-2}{2}\right\rangle$ occurs in exactly one clique of the decomposition K $(r = n(n-1)(n-2)/24)$. Here the single elements of the SQS correspond to the vertices of the graph $\left\langle n, \dfrac{n-2}{2}\right\rangle$, the quadruples of the SQS correspond to the 4-vertex-cliques K_i.

Let us now have an arbitrary 4-vertex-clique K; the length of the edges of the clique K which lie on its „circumference" correspond to the elements of the corresponding a-quadruple (see Fig. 1). From the numbers $j_1 + j_2, j_2 + j_3, j_3 + j_4, j_4 + j_1$ the two least numbers will give us the lengths of the „diagonals" of the clique. Denote these numbers by k_1, k_2. Consequently, the six numbers $j_1, j_2, j_3, j_4, k_1, k_2$ give us the lengths of the edges of the given clique K. Now let us denote by $\alpha_s(K)$, $s = 1, 2, \ldots, n/2$, the number of occurences of an edge of the length s in the clique K. Obviously we must have $\sum_{s=1}^{n/2} \alpha_s(K) = 6$. Further, each clique K contains 4 triangles: without loss of generality we may

(1) By a 4-vertex-clique we mean a subgraph with 4 vertices, where each pair of them is joined by exactly one edge.

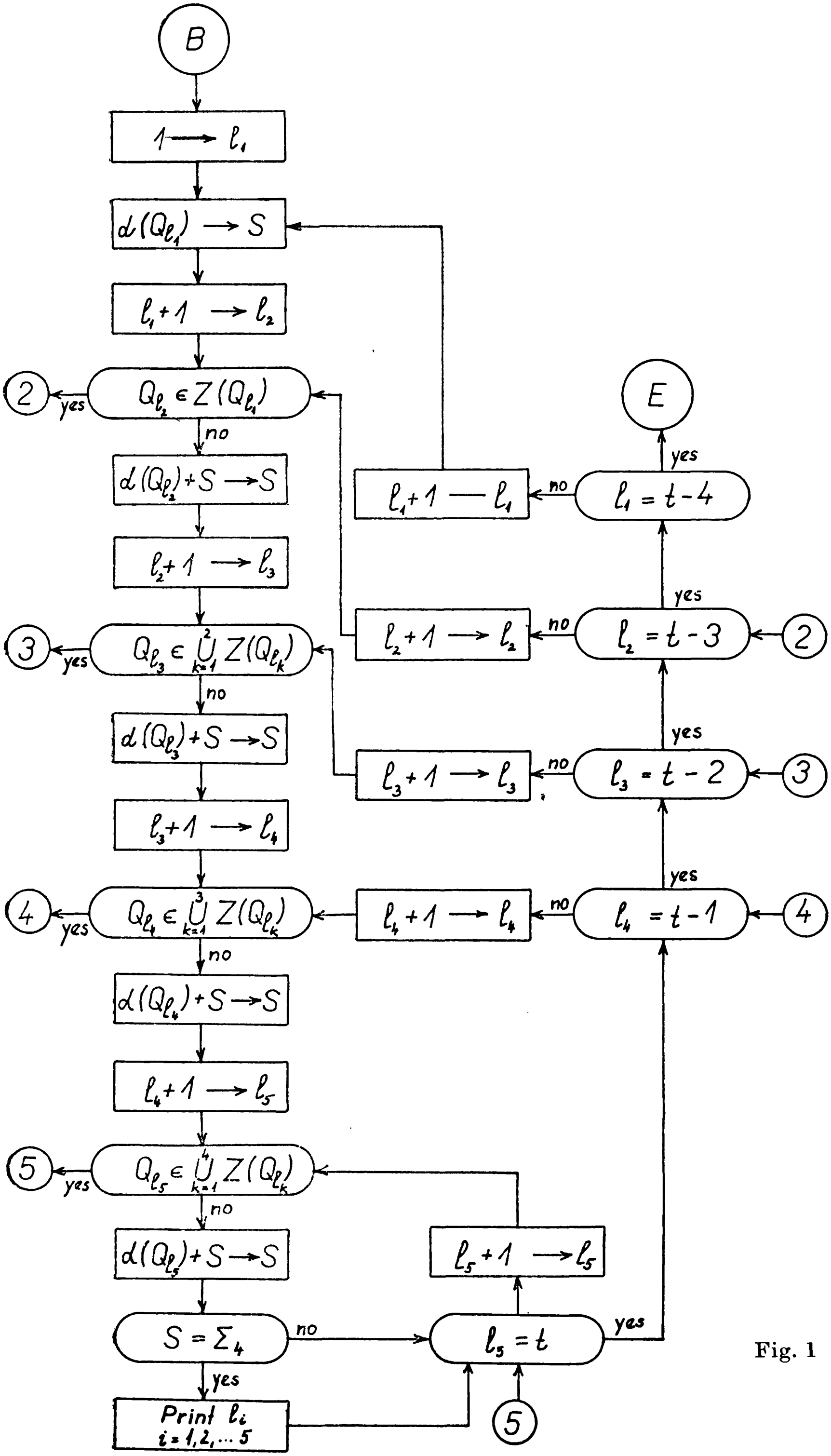
B
1 → l1
d(Ql1) → S
l1+1 → l2
Ql2 ∈ Z(Ql1)
yes
2
no
d(Ql2)+S → S
l2+1 → l3
Ql3 ∈ ∪ Z(Qlk)
k=1
3
no
d(Ql3)+S → S
l3+1 → l4
Ql4 ∈ ∪ Z(Qlk)
4
d(Ql4)+S → S
l4+1 → l5
Ql5 ∈ ∪ Z(Qlk)
5
d(Ql5)+S → S
S = Σ4
Print li
i = 1, 2, ... 5
l5 = t
l5+1 → l5
l4 = t-1
l4+1 → l4
l3 = t-2
l3+1 → l3
l2 = t-3
l2+1 → l2
l1 = t-4
l1+1 → l1
E

Fig. 1

assume them to be the triangles with the lengths of edges (in the shown order)

(IV) $$j_1j_2k_1,\ j_2j_3k_2,\ j_3j_4k_1,\ j_4j_1k_2.$$

Now let us state some evident necessary conditions for the existence of a cyclic SQS of order n which follow easily from the introduced graph-theoretical interpretation ($\alpha_s(Q)$ is defined analogously as $\alpha_s(K)$);

(α) $$\sum_Q \alpha_s(Q)\frac{\pi(Q)}{n} = \frac{n-2}{2}, \quad s = 1, 2, \ldots, \frac{n}{2} - 1:$$

$$\sum_Q \alpha_s(Q)\frac{\pi(Q)}{n} = \frac{n-2}{4}, \quad s = \frac{n}{2},$$

where the sum is extended over all different a-quadruples Q which correspond to the cyclic sets occuring in the given cyclic SQS.

(β) Let S be a cyclic SQS containing the cyclic set $M(Q)$ and let $m_1m_2m_3$ be one of the triples (IV) corresponding to the a-quadruple Q. Let Q_1, Q_2, ..., Q_t be all the remaining a-quadruples to which the same triple $m_1m_2m_3$ corresponds (besides some other three triples).

(a) If two of the numbers m_1, m_2, m_3 are equal, then S does not contain any cyclic set $M(Q_i)$, $i = 1, 2, \ldots, t$.

(b) If all three numbers m_1, m_2, m_3 are mutually different, then S contains at most one cyclic set $M(Q_i)$, $i \in \{1, 2, \ldots, t\}$.

(γ) Let the four triples $T_1 = m_1m_2m_3$, $T_2 = m_1m_2m_3$, T_3, T_4 correspond to an a-quadruple Q and let $\pi(Q) = n$. Then no cyclic SQS of order n contains the cyclic set $M(Q)$.

(0) Let the four triples

$$T_1' = m_1m_2m_3,\ T_2' = n_1n_2n_3,\ T_3',\ T_4'$$

and

$$T_1'' = m_1m_2m_3,\ T_2'' = n_1n_3n_2,\ T_3'',\ T_4''$$

correspond to a-quadruples Q' and Q'', respectively. Then each SQS of order n does not contain simultaneously the cyclic sets $M(Q')$ and $M(Q'')$.

The four given simple statements will be used in the proofs of theorems $A - D$.

Theorem A. *There exists no cyclic SQS of order* 8.

Proof. Let us suppose that there exists a cyclic SQS of order 8; it must contain 14 quadruples. All possible quadruples formed from the elements 1, 2, ..., 8 are characterized by one of the a-quadruples given in Tab. 1.

It is easy to see that the a-quadruples (1), (2), (3), (4), (6), (7) have the period 8, the a-quadruple (5) has the period 4 and the a-quadruple (8) has the period 2. Since $14 = 8 + 4 + 2$, a cyclic SQS of order 8 must necessarily contain the cyclic sets with the a-quadruples (5) and (8), i. e. together 6 quadruples, and also 8 quadruples of the cyclic set with the a-quadruple of one of the types (1), (2), (3), (4), (6), (7). However, each of these possibilities leads to a contradiction with the conditions (α) — (δ).

Table 1

(1)	[1113]		
(2)(a)	[1124]	(b)	[1142]
(4)	[1133]		
(5)	[1313]		
(6)(a)	[1223]	(b)	[1322]
(7)	[1232]		
(8)	[2222]		

Table 2

(1)	[1113]		
(2)(a)	[1124]	(b)	[1142]
(3)	[1214]		
(4)(a)	[1135]	(b)	[1153]
(5)	[1315]		
(6)	[1144]		
(7)	[1414]		
(8)(a)	[1225]	(b)	[1522]
(9)	[1252]		
(10)(a)	[1234]	(b)	[1432]
(11)(a)	[1342]	(b)	[1243]
(12)(a)	[1423]	(b)	[1324]
(13)	[1333]		
(14)	[2224]		
(15)	[2233]		
(16)	[2323]		

Theorem B. *There exists exactly one cyclic SQS of order* 10.

Proof. All possible quadruples from the elements 1, 2, ..., 10 are characterized by one of the a-quadruples given in Tab. 2. All these a-quadruples except (7) and (16) have the period 10; $\pi([1414]) = \pi([2323]) = 5$. Since a SQS of order 10 contains 30 quadruples, two cases can occur:

a) the cyclic SQS contains three cyclic sets with a-quadruples of some of the types (1)—(6), (8)—(15):

b) the cyclic SQS contains the cyclic sets with the a-quadruples (7) and (16) and two cyclic sets with the a-quadruples of the types (1)—(6), (8)—15).

Similarly as in the proof of Theorem A it can be established that the possibility b) leads to a contradiction. Between the combinations of a-quadruples (1)—(6), (8)—(15) taken three at a time in the case a) all combinations but one also lead to a contradiction. The only suitable system is given by the combination of the cyclic sets with the a-quadruples (3), (6) and (15); the corresponding cyclic SQS of order 10 has the form;

$$(i, i + 1, i + 3, i + 4) \quad (i, i + 1, i + 2, i + 6) \quad (i, i + 2, i + 4, i + 7)$$
$$i = 1, \ldots, 10 \qquad i = 1, \ldots, 10 \qquad i = 1, \ldots, 10$$

Theorem C. *There exists no cyclic SQS of order* 14.

Proof. All possible quadruples from the elements 1, 2, ..., 14 are characterized by one of the a-quadruples given in Tab. 3. All of them except (11), (38), (45) have the period 14: $\pi([1616]) = \pi[(2525)] = \pi[(3434]) = 7$. Since a SQS of order 14 contains 91 quadruples, two cases can occur:

a) the cyclic SQS consists of six cyclic sets with the a-quadruples of some of the types

$$\text{(V)} \qquad (1)-(10),\ (12)-(37),\ (39)-(44)$$

and one cyclic set with the a-quadruple from the types

$$\text{(VI)} \qquad (11),\ (38),\ (45);$$

b) the cyclic SQS consists of five cyclic sets with the a-quadruples from (V) and three cyclic sets with the a-quadruples (VI).

Further computations were made on the computer Ural-2. The cyclic sets with the a-quadruples

$$\text{(VII)} \qquad (1),\ (31),\ (32),\ (37),\ (43)$$

Table 3

(1)	[1113]		(24)	[1373]	
(2)(a)	[1124]	(b) [1142]	(25)(a)	[1346]	(b) [1643]
(3)(a)	[1135]	(b) [1153]	(26)(a)	[1355]	(b) [1553]
(4)(a)	[1146]	(b) [1164]	(27)(a)	[1364]	(b) [1463]
(5)(a)	[1157]	(b) [1175]	(28)(a)	[1436]	(b) [1634]
(6)	[1166]		(29)	[1535]	
(7)	[1214]		(30)(a)	[1445]	(b) [1544]
(8)	[1315]		(31)	[1454]	
(9)	[1416]		(32)	[2226]	
(10)	[1517]		(33)(a)	[2237]	(b) [2273]
(11)	[1616]		(23)(a)	[2246]	(b) [2264]
(12)(a)	[1225]	(b) [1522]	(35)	[2255]	
(13)	[1252]		(36)	[2327]	
(14)(a)	[1236]	(b) [1632]	(37)	[2426]	
(15)(a)	[1247]	(b) [1742]	(38)	[2525]	
(16)(a)	[1256]	(b) [1652]	(39)(a)	[2336]	(b) [2633]
(17)(a)	[1265]	(b) [1562]	(40)(a)	[2345]	(b) [2543]
(18)(a)	[1326]	(b) [1623]	(41)(a)	[2354]	(b) [2453]
(19)(a)	[1427]	(b) [1724]	(42)	[2363]	
(20)(a)	[1526]	(b) [1625]	(43)	[3335]	
(21)(a)	[1263]	(b) [1362]	(44)	[3344]	
(22)(a)	[1274]	(b) [1472]	(45)	[3434]	
(23)(a)	[1337]	(b) [1733]			

were omitted from further consideration, since according to the condition (γ) none of them can occur in a cyclic SQS of order 14. For the remaining a-quadruples, using the conditions (β) and (δ), the so-called system of prohibitions $Z = Z(Q)$ was formed, i. e., for any given a-quadruple Q the a-quadruples are given such that the cyclic sets with these a-quadruples cannot occur in a cyclic SQS of order 14 commonly with the cyclic set with the a-quadruple Q. First of all there were found on the computer (case a)) all combinations of the a-quadruples (V) minus (VII) satisfying the system of prohibitions Z and giving according to (α) in the sum of the numbers $\alpha_s(Q)$ one of three numbers Σ_1, Σ_2, Σ_3 (Tab. 4), taken six at a time. The running time was about 40 minutes.Then there were found on the computer (case b)) all combinations of the a-quadruples

(VIII) (2)—(5), (7)—(9), (12)—(14), (17), (18), (21), (26), (27), (29), (30), (34), (39), (41), (42)

satisfying the system of prohibitions Z and giving according to (α) in the sum of the numbers $\alpha_s(Q)$ the number Σ_4 (Tab. 4), taken five at a time. (The a-quadruples from (V) minus (VII) not occuring in (VIII) were omitted for the reason that cyclic sets with these a-quadruples cannot occur in any cyclic SQS containing cyclic sets with the a-quadruples (11), (38), (45).) The running time was about 15 minutes. On Fig. 2 the flow diagram of the programme in the case b) is shown (where we denoted $\alpha(Q) = (\alpha_1(Q), \ldots, \alpha_{n/2}(Q))$: the a-quadruples from (VIII) are denoted by $Q_1, Q_2, \ldots, Q_t$). The flow diagram in the case a) as well as in the cases a), b) when $n = 16$ (see below) is of an analogous form.

The existence of such combinations, whether in case a) or b), is obviously only a necessary condition for the existence of a cyclic SQS of order 14(2).

Table 4

s	1	2	3	4	5	6	7
Σ_1	5	6	6	6	6	5	2
Σ_2	6	5	6	6	5	6	2
Σ_3	6	6	5	5	6	6	2
Σ_4	5	5	5	5	5	5	0

(2) It can be shown that even the fulfilling of the conditions (α), (β), (γ), (δ) is not sufficient for the existence of a cyclic SQS. We do not mention the formulation of the necessary and sufficient condition (naturally, in terms used in the given conditions) for its complexity.

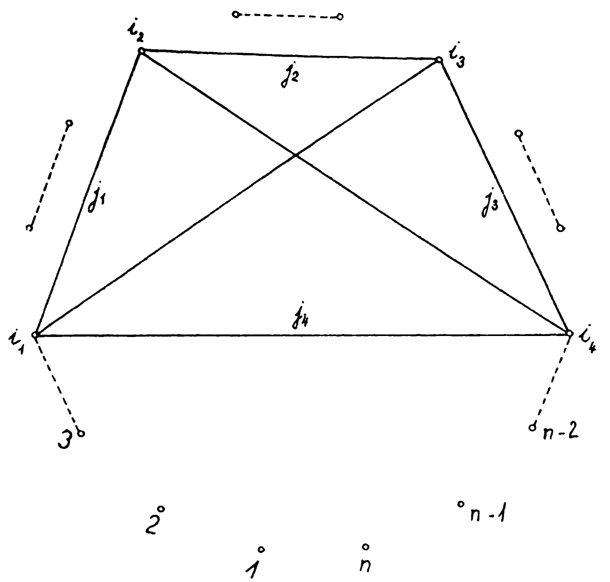

Fig. 2

It turned out that each of she possibilities (i. e., each of the printed combinations) leads to a contradiction with the definition of the SQS; consequently there does not exist a cyclic SQS of order 14.

Theorem D. *There exists no cyclic SQS of order* 16.

Proof. All possible quadruples on the elements 1, 2, ..., 16 are characterized by one of the a-quadruples given in Tab. 5. All these a-quadruples ex-

Table 5

(1) [1113]		(37) [1636]	
(2)(a) [1124]	(b) [1142]	(38)(a) [1447]	(b) [1744]
(3)(a) [1135]	(b) [1153]	(39) [1474]	
(4)(a) [1146]	(b) [1164]	(40)(a) [1456]	(b) [1654]
(5)(a) [1157]	(b) [1175]	(41)(a) [1465]	(b) [1564]
(6)(a) [1168]	(b) [1186]	(42)(a) [1546]	(b) [1645]
(7) [1177]		(43) [1555]	
(8) [1214]		(44) [2226]	
(9) [1315]		(45)(a) [2237]	(b) [2273]
(10) [1416]		(46) [2327]	
(11) [1517]		(47)(a) [2248]	(b) [2284]
(12) [1618]		(48) [2428]	
(13) [1717]		(49)(a) [2257]	(b) [2275]
(14)(a) [1225]	(b) [1522]	(50) [2527]	

(15)	[1252]			(51)	[2266]		
(16)(a)	[1236]	(b)	[1632]	(52)	[2626]		
(17)(a)	[1263]	(b)	[1362]	(53)(a)	[2338]	(b)	[2833]
(18)(a)	[1326]	(b)	[1623]	(54)	[2383]		
(19)(a)	[1247]	(b)	[1742]	(55)(a)	[2347]	(b)	[2743]
(20)(a)	[1274]	(b)	[1472]	(56)(a)	[2374]	(b)	[2473]
(21)(a)	[1427]	(b)	[1724]	(57)(a)	[2437]	(b)	[2734]
(22)(a)	[1258]	(b)	[1852]	(58)(a)	[2356]	(b)	[2653]
(23)(a)	[1285]	(b)	[1582]	(59)(a)	[2365]	(b)	[2563]
(24)(a)	[1528]	(b)	[1825]	(60)(a)	[2536]	(b)	[2635]
(25)(a)	[1267]	(b)	[1762]	(61)(a)	[2446]	(b)	[2644]
(26)(a)	[1276]	(b)	[1627]	(62)	[2464]		
(27)(a)	[1627]	(b)	[1726]	(63)(a)	[2455]	(b)	[2554]
(28)(a)	[1337]	(b)	[1733]	(64)	[2545]		
(29)	[1373]			(65)	[3337]		
(30)(a)	[1348]	(b)	[1843]	(66)(a)	[3347]	(b)	[3364]
(31)(a)	[1384]	(b)	[1483]	(67)	[3436]		
(32)(a)	[1438]	(b)	[1834]	(68)	[3355]		
(33)(a)	[1357]	(b)	[1753]	(69)	[3535]		
(34)(a)	[1375]	(b)	[1573]	(70)(a)	[3445]	(b)	[3544]
(35)(a)	[1537]	(b)	[1735]	(71)	[3454]		
(36)(a)	[1366]	(b)	[1663]	(72)	[4444]		

cept (13), (52), (69) and (72) have the period 16; $\pi([1717]) = \pi([2626]) = \pi([3535]) = 8$; $\pi([4444]) = 4$. Since a SQS of order 16 consists of 140 quadruples, it follows from (III) that two cases can occur:

a) the cyclic SQS consists of eight cyclic sets with a-quadruples of some of the types

(IX) (1)—(12), (14)—(51), (53)—(68), (70), (71),

of one cyclic set with an a-quadruple of one of the types

(X) (13), (52), (69)

and the cyclic set with the a-quadruple (72);

b) the cyclic SQS consists of seven cyclic sets with the a-quadruples from (IX), three cyclic sets with the a-quadruples (X) and the cyclic set with the a-quadruple (72).

In both cases the cyclic SQS of order 16 must contain the cyclic set with the a-quadruple (72), therefore according to the condition (β), the a-quadruples

(XI) (30), (31), (38) ,(47), (61), (70)

could be omitted from further considerations. Besides them according to (γ) also the a-quadruples

(XII) (1), (43), (44), (50), (62), (65)

could be omitted. For the remaining a-quadruples, using the conditions (β) and (γ), the system of prohibitions Z was formed and further computations on the computer Ural-2 were made entirely analogously as in the case of $n = 14$. In the case a) all combinations — taken eight at a time — of a-quadruples (IX) were found (from whose the a-quadruples (XI) and (XII) were eliminated) satisfying the system of prohibitions Z and giving according to (α) in the sum of the numbers $\alpha_s(Q)$ one of the numbers Ω_1, Ω_2, Ω_3 (Tab. 6). In the case b) all combinations — taken seven at a time — of the corresponding a-quadruples satisfying the system of prohibitions Z and giving according to (α) in the sum of the numbers $\alpha_s(Q)$ the number Ω_4 (Tab. 6) were found. The running time was 110 minutes in the case a) and about 45 minutes in the case b). Checking of the printed combinations in both cases showed that there does not exist a cyclic SQS of order 16.

Table 6

s	1	2	3	4	5	6	7	8
Ω_1	6	7	7	6	7	7	6	2
Ω_2	7	6	7	6	7	6	7	2
Ω_3	7	7	6	6	6	7	7	2
Ω_4	6	6	6	6	6	6	6	0

Remark. To the construction of cyclic SQS of order n, where $n \equiv 2$ or 10 (mod 24), a separate paper will be devoted. Here we bring only as an example one cyclic SQS of order 26.

Example. A cyclic SQS of order 26 is formed by the following quadruples ($i = 1, 2, \ldots, 26$; the numbers in quadruples are taken modulo 26);

$(i, i+1, i+2, i+14)$ $(i, i+1, i+9, i+10)$ $(i, i+5, i+11, i+16)$
$(i, i+2, i+4, i+15)$ $(i, i+1, i+11, i+12)$ $(i, i+7, i+9, i+16)$
$(i, i+3, i+6, i+16)$ $(i, i+3, i+5, i+8)$ $(i, i+7, i+11, i+18)$
$(i, i+4, i+8, i+17)$ $(i, i+3, i+7, i+10)$ $(i, i+9, i+11, i+20)$
$(i, i+5, i+10, i+18)$ $(i, i+3, i+9, i+12)$ $(i, i+2, i+8, i+10)$
$(i, i+6, i+12, i+19)$ $(i, i+3, i+11, i+14)$ $(i, i+4, i+6, i+10)$
$(i, i+1, i+3, i+4)$ $(i, i+5, i+7, i+12)$ $(i, i+8, i+12, i+20)$
$(i, i+1, i+5, i+6)$ $(i, i+5, i+9, i+14)$ $(i, i+10, i+12, i+22)$
$(i, i+1, i+7, i+8)$

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