Matematický časopis

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Matematický časopis, Vol. 21 (1971), No. 2, 106--109

Persistent URL: http://dml.cz/dmlcz/126437

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ON A CLASS OF DARBOUX FUNCTIONS FROM A TOPOLOGICAL SPACE TO A UNIFORM SPACE

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Let X be a topological space, Y be a uniform space, $\mathscr{B}$ be a base of open connected sets in X, $\mathscr{M}$ be a base of the uniformity of Y. We shall say that a function $f: X \to Y$ belongs to $D_0''(\mathscr{B})$ if and only if there are no $U \in \mathscr{B}$, $V \in \mathscr{M}$, A, $B \subset Y$ such that $\overline{f(\overline{U})} = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$, $A \times B \subset Y \times Y \setminus V$. The family $D_0''(\mathscr{B})$ does not depend on the choice of a base $\mathscr{M}$ of the uniformity of Y. If Y is a metric space then $f \in D_0(\mathscr{B})$ means that for any $U \in \mathscr{B}$, $\overline{f(\overline{U})}$ cannot be written as a union of two non-empty sets with a positive distance.

We prove first that $D_0''(\mathscr{B})$ is closed under the limits of uniformly convergent sequences. If Y is moreover an abelian topological group, X is regular and $\mathscr{B}$ fulfils an additional condition (1*) (see Lemma 2), then $f + g \in D_0''(\mathscr{B})$ for any $f, g \in D_0''(\mathscr{B})$ such that in any point $x \in X$ at least one of f, g is continuous.

The families $D(\mathscr{B})$ (resp. $D_0(\mathscr{B})$) of all real-valued functions with the Darboux property (resp. with the Darboux property in the Radakovitch sense) on a topological space were introduced and studied by L. Mišík ([3], [4]). In [1] J. Farková introduced two similar families $D_0'(\mathscr{B})$, $D'(\mathscr{B})$ from a topological space X to a metric space Y. By Farková's definition, $f \in D_0'(\mathscr{B})$ if and only if $\overline{f(\overline{U})}$ is connected for any $U \in \mathscr{B}$.

Clearly $D_0'(\mathscr{B}) = D_0''(\mathscr{B})$ if Y is the real line. But in the general case (when only $D_0'(\mathscr{B}) \subset D_0''(\mathscr{B})$) the family $D_0''(\mathscr{B})$ seems to be more convenient since for $D_0'(\mathscr{B})$ the above mentioned theorems do not hold.*) Of course, our family $D_0''(\mathscr{B})$ has a meaning only if the range space Y is uniform. In a certain sense we extend Farková's results in two directions: we consider a larger class of range spaces Y and a larger class of functions $D_0''(\mathscr{B})$.

Theorem 1. *Let $\{f_n\}$ be a sequence of functions belonging to $D_0''(\mathscr{B})$ and converging uniformly on X to a function f. Then $f \in D_0''(\mathscr{B})$.*

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*) The corresponding results of Farková contain some additional assumptions and follow from our theorems.

Proof. Assume that $f \notin D_0''(\mathscr{B})$. Then there are $A, B \subset Y$, $U \in \mathscr{B}$, $V \in \mathscr{M}$ such that $A \times B \subset Y \times Y \setminus V$, $\overline{f(\overline{U})} = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$. Put $C = \{x \in \overline{U} : f(x) \in A\}$, $D = \{x \in \overline{U} : f(x) \in B\}$. For $S, T \in \mathscr{M}$ denote by $S \circ T$, as usually, the set $\{(x, y)$: there is $z \in Y$ such that $(x, z) \in S$, $(z, y) \in T\}$. Then there is $W \in \mathscr{M}$ such that $W \circ W \circ W \subset V$ (see [2], chapter 6). Choose n such that $(f_n(x), f(x)) \in W$ for all $x \in X$.

Clearly $C \neq \emptyset$, $D \neq \emptyset$. We assert that $f_n(C) \times f_n(D) \subset Y \times Y \setminus W$. In the reverse case there are $x \in C$, $y \in D$ such that $(f_n(x), f_n(y)) \in W$. But then $(f(x), f_n(x)) \in W$, $(f_n(x), f_n(y)) \in W$, $(f_n(y), f(y)) \in W$ and $(f(x), f(y)) \in W \circ W \circ W \subset V$. But $f(x) \in A$, $f(y) \in B$, hence $A \times B \cap V \neq \emptyset$, which is a contradiction with the property of V stated above.

Hence $f_n(C) \times f_n(D) \subset Y \times Y \setminus W$ for a sufficiently large n, which is a contradiction with the assumption $f_n \in D_0''(\mathscr{B})$.

The corresponding result of Farková follows from Theorem 1 and the following lemma.

Lemma 1. *If $f \in D_0''(\mathscr{B})$ and there is a compact set C such that $f(X) \subset C$, then $f \in D_0'(\mathscr{B})$.*

Proof. If $f \notin D_0'(\mathscr{B})$, then $\overline{f(\overline{U})} = A \cup B$, where A, B are disjoint, non-void and moreover compact. Hence there is $V \in \mathscr{M}$ such that $A \times B \subset Y \times Y \setminus V$, therefore $f \notin D_0''(\mathscr{B})$.

Corrolary ([1], Theorem 1). *If $\{f_n\}$ converges uniformly to f, $f_n \in D_0'(\mathscr{B})$ ($n = 1, 2, \ldots$) and there is a compact set C such that $f(X) \subset C$, then $f \in D_0'(\mathscr{B})$.*

Proof. Clearly $D_0'(\mathscr{B}) \subset D_0''(\mathscr{B})$. Then $f \in D_0''(\mathscr{B})$, according to Theorem 1 and $f \in D_0'(\mathscr{B})$ according to Lemma 1.

Let Y be now an abelian topological group. It is well known that Y becomes a uniform space in which the family of all sets of the form $\{(x, y) : x - y \in W\}$, where W is an open neighbourhood of the zero element O, is a base of the uniformity. Hence $f \in D_0''(\mathscr{B})$ if and only if there are no $U \in \mathscr{B}$, $A, B \subset Y$ and no neighbourhood W of O such that $A - B \subset Y \setminus W$,*) $\overline{f(\overline{U})} = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$.

In the following we shall use the following lemma due to J. Farková in [1].

Lemma 2. *Let X be a topological locally connected space, $\mathscr{B}$ be a base of open connected sets satisfying the following property*:

(1*) *To any open F, any $E \in \mathscr{B}$ and any $x \in F \cap \overline{E}$ there is $C \in \mathscr{B}$ such that $C \subset F \cap E$, $x \in \overline{C}$.*

Let $F \in \mathscr{B}$, $\overline{F} = C \cup D$, where C, D are disjoint non-void sets.

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*) While $A \setminus B$ means the set theoretic difference, $A - B = \{u : u = x - y,\ x \in A,\ y \in B\}$.

Then there is $x_0 \in \bar{C} \cap \bar{D}$ such that to any neighbourhood V of x_0 there is $U \in \mathscr{B}$, $U \subset V$, $x_0 \in \bar{U}$, $\bar{U} = (\bar{U} \cap C) \cup (\bar{U} \cap D)$, and $\bar{U} \cap C$, $\bar{U} \cap D$ are disjoint non-void sets.

Theorem 2. *Let X be a regular topological space, $\mathscr{B}$ a base consisting of open connected sets fulfilling (1*). Let Y be an abelian topological group, $f, g \in D_0''(\mathscr{B})$ and any $x \in X$ is a point of continuity of at least one of the functions f, g. Then $f + g \in D_0''(\mathscr{B})$.*

Proof. Let $f + g \notin D_0''(\mathscr{B})$. Then there are $F \in \mathscr{B}$, $A, B \subset Y$ and a neighbourhood W of O such that $A - B \subset Y \setminus W$, $\overline{f + g(\bar{F})} = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$. Let T be such a symmetric neighbourhood of O that $T + T + T + T \subset W$.

Put $C = \bar{F} \cap (f + g)^{-1}(A)$, $D = \bar{F} \cap (f + g)^{-1}(B)$. Then $C \cap D = \emptyset$, C, D are non-void and $\bar{F} = C \cup D$. Let x_0 be an element of $\bar{C} \cap \bar{D}$ having the properties stated in Lemma 2. Let e. g. f be continuous in x_0. As X is regular, there is an open neighbourhood V of x_0 such that $f(u) - f(x_0) \in T$ for all $u \in \bar{V}$. Finally, let U have the properties stated in Lemma 2 with respect to this set V.

Let $x \in \bar{U} \cap C$, $y \in \bar{U} \cap D$. Then $f + g(x) - f + g(y) \notin W$ but $f(x) - f(y) = f(x) - f(x_0) + f(x_0) - f(y) \in T + T$. If $g(x) - g(y) \in T + T$ then $f + g(x) - f + g(y) = f(x) - f(y) + g(x) - g(y) \in T + T + T + T \subset W$, which is impossible. Therefore $g(x) - g(y) \notin T + T$ for all $x \in \bar{U} \cap C$, $y \in \bar{U} \cap D$, or by others words $g(\bar{U}) \cap C) - g(\bar{U} \cap D) \subset Y \setminus (T + T)$.

For $K, L \subset Y$, $K \neq \emptyset$, $L \neq \emptyset$ write $\varrho(K, L) > 0$, whenever there is $S \in \mathscr{M}$ such that $K \times L \subset Y \times Y \setminus S$. We have just proved $\varrho(g(\bar{U} \cap C), g(\bar{U} \cap D)) > 0$. The proof will be complete if we prove that $\varrho(K, L) > 0$ implies $\varrho(\bar{K}, \bar{L}) > 0$. Indeed, we get $\varrho(\overline{g(\bar{U} \cap C)}, \overline{g(\bar{U} \cap D)}) > 0$, hence $\overline{g(\bar{U})} = \overline{g(\bar{U} \cap C)} \cup \overline{g(\bar{U} \cap D)}$, where $\overline{g(\bar{U} \cap C)}$, $\overline{g(\bar{U} \cap D)}$ are nonvoid disjoint sets of "positive distance" therefore $g \notin D_0''(\mathscr{B})$.

But the implication $\varrho(K, L) > 0 \Rightarrow \varrho(\bar{K}, \bar{L}) > 0$ can be proved easily as an exercise. Indeed, $\varrho(K, L) > 0$ implies the existence of a neighbourhood Z of O such that $K - L \subset Y \setminus Z$. Take a symmetric neighbourhood R of O such that $R + R + R \subset Z$. We prove $\bar{K} - \bar{L} \subset Y \setminus R$. In the reverse case there are $x \in \bar{K}$, $y \in \bar{L}$ such that $x - y \in R$. Then also there are $u \in K$, $v \in L$ such that $u \in R + y$, $v \in R + x$. Therefore $u - v \in R + R + R \subset Z$, which is a contradiction to the inclusion $K - L \subset Y \setminus Z$.

By proving the last implication also the proof of Theorem 2 is complete.

Corrolary 1 ([1], Theorem 2). *Let X be a regular topological space, $\mathscr{B}$ a base of open connected sets fulfilling the condition (1*). Let Y be the real line. Let $f, g \in D_0'(\mathscr{B})$ and any point of X is a point of continuity of either of g. Then $f + g \in D_0'(\mathscr{B})$.*

Proof follows immediately from Theorem 2, because $D_0'(\mathscr{B}) = D_0''(\mathscr{B})$ in this case.

Corrolary 2 ([1], Theorem 3). *Let* X, $\mathscr{B}$ *fulfil the assumptions of the previous Corrolary. Let* Y *be a linear metric space. Let* $f, g \in D_0'(\mathscr{B})$ *and any point of* X *is a point of continuity of either* f *or* g. *Let there exist a compact set* C *such that* $f + g(X) \subset C$. *Then* $f + g \in D_0'(\mathscr{B})$.

Proof follows from Theorem 2 and Lemma 1.

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Received June 12, 1969

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