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ON THE PRODUCT OF VECTOR MEASURES WITH VALUES IN SEMIORDERED SPACES

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There are several papers by M. Duchoň (resp. I. Kluvánek) devoted to the study of the product of vector-valued measures (see [1], [2], [3], [4]). Here we should like only to present some ideas or concepts concerning this object. We study measures with values in linear lattices (especially the so-called regular K-spaces) and we present two methods.

A linear lattice X is called a regular K-space (see [5], [6]) if it is conditionally complete and if for any sequence $\{\{a_n^i\}_{n=1}^\infty\}_{i=1}^\infty$ of convergent (to an a^i) sequences there is a common regulator of convergence u, i. e. to any number $\delta > 0$ and any i there is N_i such that $|a_n^i - a^i| < \delta u$ for any $n > N_i$. (A very simple example of a regular K-space is the space of all measurable functions on $\langle a, b \rangle$.)

Finally some fixed notations: $(S, \mathscr{S})$, $(T, \mathscr{T})$ are given measurable spaces, $\mathscr{D} = \{E \times F : E \in \mathscr{S},\ F \in \mathscr{T}\}$, and $\mathscr{R}$, resp. $\mathscr{S} \times \mathscr{T}$, is the ring, resp. σ-ring, generated by $\mathscr{D}$.

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Let X, Y, Z be linear lattices (K-lineals), π be a mapping $\pi: X \times Y \to Z$ satisfying the following conditions:

1. $\pi(a + b, c) = \pi(a, c) + \pi(b, c)$ for all $a, b \in X, c \in Y$,
   $\pi(a, b + c) = \pi(a, b) + \pi(a, c)$ for all $a \in X, b, c \in Y$.
2. If $O \leqq a$, $O \leqq b$, $a \in X$, $b \in Y$, then $O \leqq \pi(a, b)$.
3. If $O \leqq a_n \nearrow a$, $O \leqq b_n \nearrow b$ (resp. $a_n \searrow a$, $b_n \searrow b$), $a_n, a \in X, b_n, b \in Y$, then $\pi(a_n, b_n) \nearrow \pi(a, b)$ (resp. $\pi(a_n, b_n) \searrow \pi(a, b)$).

We shall have two positive measures α, β with values in X, resp. Y, $\alpha : \mathscr{S} \to X$, $\beta : \mathscr{T} \to Y$. And we shall construct a measure γ on $\mathscr{S} \times \mathscr{T}$ such that $\gamma(E \times F) = \pi(\alpha(E), \beta(F))$ for any $E \in \mathscr{S}$, $F \in \mathscr{T}$. Sometimes we shall admit an ideal element ∞ as a possible value of α, β, γ. In the case we shall write e. g. $\alpha : \mathscr{S} \to X^*$. If $\alpha : \mathscr{S} \to X$ (i. e. $\alpha(E) \neq \infty$ $\alpha(E) \in X$ for any $E \in \mathscr{S}$), we say also that α is a finite measure.

Lemma 1. *For any* $E \in \mathscr{S}$, $F \in \mathscr{T}$ *put* $\gamma(E \times F) = \pi(\alpha(E), \beta(F))$. *Then* $\gamma : \mathscr{D} \to Z$ *is an additive set function.*

Lemma 2. *Let* $A = \bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{m} B_j$, A_i, *resp.* B_j *be pairwise disjoint,* $A_i \in \mathscr{D}$, $B_j \in \mathscr{D}$. *Then*

$$\sum_{i=1}^{n} \gamma(A_i) = \sum_{j=1}^{m} \gamma(B_j) .$$

Proofs of Lemmas 1 and 2 can be obtained similarly as for scalar measures and therefore we omit them. Note only that Lemmas 1 and 2 hold even if X, Y and Z are arbitrary abelian groups and $\pi : X \times Y \to Z$ satisfies 1.

Definition 1. *For* $E \times F \in \mathscr{D}$ *we define* $\gamma(E \times F) = \pi(\alpha(E), \beta(F))$. *For* $A \in \mathscr{R}$, $A = \bigcup_{i=1}^{m} A_i$, $A_i \in \mathscr{D}$, A_i *pairwise disjoint we define* $\gamma(A) = \sum_{i=1}^{m} \gamma(A_i)$.

Now we must make some further assumptions concerning α and β.

Definition 2. *Let* S *be a topological space,* $\mathscr{C}$ *be a system of compact subsets of* S, $\mathscr{U}$ *be a system of open subsets of* S, $\mathscr{C} \cup \mathscr{U} \subset \mathscr{S}$. *A function* $\alpha : \mathscr{S} \to X$ *is called regular if to any* $E \in \mathscr{S}$ *there is a non-decreasing sequence* $\{C_n\}$ *of sets of* $\mathscr{C}$ *and a non-increasing sequence* $\{U_n\}$ *of sets of* $\mathscr{U}$ *such that*

$$\alpha(E) = \lim \alpha(C_n) = \lim \alpha(U_n) .$$

Theorem 1. *If* α, β *are regular finite positive measures and* Z *is a regular* K*-space then* γ *is* σ*-additive on* $\mathscr{D}$.

Proof. Let $A = \bigcup_{n=1}^{\infty} A_n$, $A \in \mathscr{D}$, $A_n \in \mathscr{D}$, A_n pairwise disjoint, $A = E \times F$, $A_n = E_n \times F_n$. According to the regularity of α and β there are sequences $\{C_i\}$, $\{D_i\}$ belonging to corresponding systems of compact sets such that

$$C_i \nearrow E, D_i \nearrow F, \alpha(C_i) \nearrow \alpha(E), \beta(D_i) \nearrow \beta(F).$$

Hence according to the axiom 3

$$\gamma(C_i \times D_i) \nearrow \gamma(E \times F) .$$

Similarly choose U_i^n, V_i^n such shat

$$U_i^n \searrow E_n, V_i^u \searrow F_n, \gamma(U_i^n \times V_i^n) \searrow \gamma(E_n \times F_n) \quad (i \to \infty) .$$

Let u be a common regulator of convergence of all the sequences $\{\gamma(U_i^n \times V_i^n)\}_{i=1}^{\infty}$, $(n = 1, 2, \ldots)$, $\{\gamma(C \times D_i)\}_{i=1}^{\infty}$. Then to any number $\delta > 0$ there is i_0 such that $\gamma(E \times F) - \gamma(C_{i_0} \times D_{i_0}) < \delta/2\, u$.

Further there is $i(n)$ such that

$$\gamma(U^n_{i(n)} \times V^n_{i(n)}) - \gamma(E_n \times F_n) < \frac{\delta}{2^{n+1}} u .$$

Put $U_n = U^n_{i(n)}$, $V_n = V^n_{i(n)}$, $C = C_{i_0}$, $D = D_{i_0}$. Then

$$C \times D \subset E \times F = \bigcup_{n=1}^{\infty} E_n \times F_n \subset \bigcup_{n=1}^{\infty} U_n \times V_n , \tag{1}$$

$$\gamma(E \times F) - \gamma(C \times D) < \frac{\delta}{2} u , \tag{2}$$

and

$$\gamma(U_n \times V_n) - \gamma(E_n \times F_n) < \frac{\delta}{2^{n+1}} u, \quad (n = 1, 2, \ldots). \tag{3}$$

Since $C \times D$ is compact, $U_n \times V_n$ open $(n = 1, 2, \ldots)$ we get from (1) that there is N with

$$C \times D \subset \bigcup_{n=1}^{N} U_n \times V_n .$$

From the additivity of γ the subadditivity follows, hence

$$\gamma(C \times D) \leqq \sum_{n=1}^{N} \gamma(U_n \times V_n) . \tag{4}$$

Now recall another consequence of the additivity of γ:

$$\gamma(E \times F) \geqq \sum_{n=1}^{\infty} \gamma(E_n \times F_n) . \tag{5}$$

According to (2), (3), (4) and with regard to (5) we have

$$\gamma(E \times F) < \gamma(C \times D) + \frac{\delta}{2} u \leqq \sum_{n=1}^{N} \gamma(U_n \times V_n) + \frac{\delta}{2} u <$$

$$< \sum_{n=1}^{N} \gamma(E_n \times F_n) + \left(\sum_{n=1}^{N} \frac{\delta}{2^{n+1}}\right) u + \frac{\delta}{2} u \leqq$$

$$\leqq \sum_{n=1}^{\infty} \gamma(E_n \times F_n) + \left(\sum_{n=1}^{N+1} \frac{\delta}{2^n}\right) u \leqq \sum_{n=1}^{\infty} \gamma(E_n \times F_n) + \delta u .$$

From the last inequality we obtain

$$\gamma(E \times F) \leqq \sum_{n=1}^{\infty} \gamma(E_n \times F_n) ,$$

hence according to (5) also

$$\gamma(E \times F) = \sum_{n=1}^{\infty} \gamma(E_n \times F_n) .$$

Theorem 2. *If γ is σ-additive on $\mathscr{D}$ then γ is σ-additive on $\mathscr{R}$ (Z being arbitrary).*

Proof. Let $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \in \mathscr{R}$, $A \in \mathscr{R}$, A_i pairwise disjoint, $A = \bigcup_{j=1}^{m} B_j$, B_j disjoint, $B_j \in \mathscr{D}$, $A_i = \bigcup_{n=1}^{k_i} A_i^n$, $A_i^n \in \mathscr{D}$, A_i^n disjoint.

Then

$$\gamma(A) = \sum \gamma(B_j) = \sum_{j=1}^{m} \sum_{i=1}^{\infty} \sum_{n=1}^{k_i} \gamma(A_i^n \cap B_j) = \sum_{i=1}^{\infty} \sum_{j=1}^{m} \sum_{n=1}^{k_i} \gamma(A_i^n \cap B_j) = \sum_{i=1}^{\infty} \gamma(A_i) .$$

Lemma 3. *Let $\mathscr{C}$ (resp. $\mathscr{U}$) be closed under countable intersections (resp. unions) and finite unions (resp. intersections). Let τ be a positive finite measure with values in a regular K-space. If $\{E_n\}_{n=1}^{\infty}$ is a monotone sequence of regular sets, then $\lim E_n$ is also regular.*

Proof. We prove the assertion for descending sequences. If $E_n \nearrow E$, E_n are regular ($n = 1, 2, \ldots$), then there are C_n^m compact, U_n^m open such that $C_n^m \subset C_n^{m+1}$, $U_n^m \subset U_n^{m+1}$ ($m = 1, 2, \ldots$) and $\tau(E_n) = \lim \tau(C_n^m) = \lim \tau(U_n^m)$.

Let u be a common regulator of convergence of all $\{\tau(C_n^m)\}_{m=1}^{\infty}$, all $\{\tau(U_n^m)\}_{m=1}^{\infty}$ and $\{\tau(E_n)\}_{n=1}^{\infty}$. Then to any positive integer k there is such an $n = n(k)$ that $\tau(E) - \tau(E_n) < (1/k)u$ and to the n there is such an m that $\tau(E_n) - \tau(C_n^m) < (1/k)u$. Now if we denote the set C_n^m by C_k and put $D_j = \bigcup_{i=1}^{j} C_i$ ($j = 1, 2, \ldots$), we obtain a sequence $\{D_k\}_{k=1}^{\infty}$ of compact sets such that $D_i \subset D_{i+1}$ ($i = 1, 2, \ldots$) and $\tau(E) = \lim \tau(D_k)$.

On the other hand choose $U_n = U_n^m$ such that $\tau(U_n) - \tau(E_n) < (\delta 2^{-n})u$. Then $U = \bigcup_{n=1}^{\infty} U_n \supset \bigcup_{n=1}^{\infty} E_n = E$ and $\tau(U) - \tau(E) \leqq \sum_{n=1}^{\infty} (\tau(U_n) - \tau(E_n)) \leqq \delta u$.

Theorem 3. *Let α, β be regular finite positive measures, Z be a regular K-space. Then there is just one positive measure $\gamma\colon \mathscr{S} \times \mathscr{T} \to Z$ such that*

$$\gamma(E \times F) = \pi(\alpha(E), \beta(F))$$

for any $E \in \mathscr{S}$, $F \in \mathscr{T}$. If $\mathscr{S}$, $\mathscr{T}$ are σ-algebras, and $\mathscr{C}$ (resp. $\mathscr{U}$) is closed under *countable intersections (resp. unions) and finite unions (resp. intersections), then the measure γ is regular.*

Proof. Let γ be the function $\gamma : \mathscr{R} \to Z$ defined in Definition 1. Then γ is a measure according to Theorem 1 and Theorem 2. According to [7], Theorem 11, there is just one extension (denote it by the same letter γ) of γ to $\mathscr{S} \times \mathscr{T}$, which is a measure. Hence the existence is proved.

If τ is another measure on $\mathscr{S} \times \mathscr{T}$, identical with γ on $\mathscr{D}$ (i. e. $\tau(E \times F) = \gamma(E \times F) = \pi(\alpha(E), \beta(F))$, then evidently $\tau = \gamma$ on $\mathscr{R}$ and therefore $\tau = \gamma$ according to [7], Theorem 11.

Finally we prove that γ is regular assuming $\mathscr{S}$, $\mathscr{T}$ algebras. γ is evidently regular on $\mathscr{R}$. Denote by $\mathscr{K}$ the family of all regular sets. Then $\mathscr{K} \supset \mathscr{R}$ and $\mathscr{K}$ is a monotone family according so Lemma 3. Hence $\mathscr{K} \supset \mathscr{S} \times \mathscr{T}$.

Examples: 1. $X = Y = Z = (-\infty, \infty)$, $\pi(x, y) = xy$. 2. X, Y any regular K-spaces, $Z = X \times Y$, $(x, y) \leqq (u, v) \Leftrightarrow x \leqq u$ and $y \leqq v$; $\pi(x, y) = (x, y)$.

Theorem 4. *Every finite, positive vector-valued Baire measure γ in a locally compact Hausdorff space is regular.*

Proof. Denote by $\mathscr{O}$ the family of all regular sets, by $\mathscr{C}$ the family of all compact G_δ sets. Evidently $\mathscr{C} \subset \mathscr{O}$. The fact that $\mathscr{O}$ is a ring follows from the following property: If $C \subset E \subset U$, $D \subset F \subset V$, then

$$C \cup D \subset E \cup F \subset U \cup V,\ (U \cup V) - (E \cup F) \subset (U - E) \cup (V - F),$$
$$(E \cup F) - (C \cup D) \subset (E - C) \cup (F - D)$$

and

$$C - V \subset E - F \subset U - D,\ (U - D) - (E - F) \subset (U - E) \cup (F - D),$$
$$(E - F) - (C - V) \subset (E - C) \cup (V - F).$$

Finally $\mathscr{O}$ is a σ-ring according to Lemma 3. Hence $\mathscr{O}$ contains all Baire sets.

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Now we shall write $\pi(x, y) = xy$ and we shall explicitely assume only that $\pi : X \times Y \to Z$. Pet $(S, \mathscr{S})$ be a measurable space $\alpha : \mathscr{S} \to X$ be a vector-valued measure. We shall assume to have „a convenient integration theory", i. e. a set $\mathscr{F}$ of integrable functions $f : S \to Y$ and an integral $J(f) = \int f \, d\alpha$ for $f \in \mathscr{F}$, fulfilling some properties.

Definition 3. *Let $\mathscr{F}$ be a family of functions $f : S \to Y$ and J be a function $J : \mathscr{F} \to Z$ satisfying the following conditions:*

1. *If f is simple, $f = \sum_{i=1}^{n} c_i \chi_{E_i}$, then $f \in \mathscr{F}$, $J(f) = \sum_{i=1}^{n} c_i \alpha(E_i)$.*

2. *If $f \geqq 0, f \in \mathscr{F}$, then $J(f) \geqq 0$.*

3. *If $f_n \geqq 0, f_n \in \mathscr{F}$ $(n = 1, 2, \ldots)$ and $f_n \nearrow f$ (resp. $f_n \searrow f$) $\{J(f_n)\}$ is bounded, then $f \in \mathscr{F}$ and $J(f_n) \to J(f)$.*

4. *$J(f + g) = J(f) + J(g)$ for any $f, g \in \mathscr{F}$.*

Under these assumptions we can construct a product of any two vector-valued measures $\alpha : \mathscr{S} \to X, \beta : \mathscr{T} \to Y$ as a measure with values in Z. We shall write also $J(f) = \int f \, d\alpha = \int f(x) \, d\alpha(x)$.

Theorem 5. *Let $(S, \mathscr{S})$, $(T, \mathscr{T})$ be measurable spaces, $\mathscr{T}$ be a σ-algebra, α, β be positive vector-valued measures, $\alpha : \mathscr{S} \to X$, $\beta : \mathscr{T} \to Y$, β be finite. Then there is just one vector-valued measure $\gamma : \mathscr{S} \times \mathscr{T} \to Z$ such that $\gamma(E \times F) = \alpha(E)\beta(F)$ for all $E \in \mathscr{S}$, $F \in \mathscr{T}$.*

Proof. For $A \in \mathscr{S} \times \mathscr{T}$ and $x \in S$ put $A^x = \{y : (x, y) \in A\}$ and $f_A(x) = \beta(A^x)$. Evidently $f_A : S \to Y$. First we prove thas $f_A \in \mathscr{F}$. Put

$$\mathscr{K} = \{A \in \mathscr{S} \times \mathscr{T} : f_A \in \mathscr{F}\} .$$

If $A = E \times F$, $E \in \mathscr{S}$, $F \in \mathscr{T}$, then $f_A = \chi_E\beta(F)$ and $f_A \in \mathscr{F}$. If $A \in \mathscr{R}$, $A = \cup A_i$, $A_i \in \mathscr{D}$, A_i disjoint, $f_A = \sum f_{A_i} \in \mathscr{F}$. Hence we see that $\mathscr{R} \subset \mathscr{K}$. $\mathscr{K}$ is a monotone system according to the Axiom 3, hence $\mathscr{K} \supset \mathscr{S} \times \mathscr{T}$.

Now we can define a function $\gamma : \mathscr{S} \times \mathscr{T} \to Z$ by the equality

$$\gamma(A) = \int \beta(A^x) \, d\alpha(x) \ \ (= J(f_A)) .$$

γ is a measure by the axioms 3 and 4. Further for $E \in \mathscr{S}$, $F \in \mathscr{T}$ we obtain

$$\gamma(E \times F) = \int \beta((E \times F)^x) \, d\alpha(x) = \int \chi_E\beta(F) \, d\alpha = \alpha(E)\beta(F) .$$

Let τ be any vector-valued measure $\tau : \mathscr{S} \times \mathscr{T} \to Z$ such that $\tau(E \times F) = \alpha(E)\beta(F)$, i. e. $\tau(A) = \gamma(A)$for $A \in \mathscr{D}$. Then also $\tau(A) = \gamma(A)$ for $A \in \mathscr{R}$. The family $\mathscr{L} = \{A \in \mathscr{S} \times \mathscr{T} : \gamma(A) = \tau(A)\}$ is monotone, hence $\gamma = \tau$ on $\mathscr{M}(\mathscr{R}) = \mathscr{S}(\mathscr{R}) = \mathscr{S} \times \mathscr{T}$.

Now we shall present an example of a ,,convenient integration theory".

Theorem 6. *Let X be a regular K-space, $(S, \mathscr{S})$, $(T, \mathscr{T})$ be measurable spaces $\mathscr{T}$, $\mathscr{S}$ be σ-algebras. Let $\alpha : \mathscr{S} \to X$ be a positive finite vector-valued measure, $\beta : \mathscr{T} \to R$ a positive real-valued measure. Then there is just one measure $\gamma : \mathscr{S} \times \mathscr{T} \to X$ such that $\gamma(E \times F) = \beta(F)\, \alpha(E)$ for every $E \in \mathscr{S}$, $F \in \mathscr{T}$.*

Proof. We want to apply Theorem 5. Here $Z = X$, $Y = R$. We must only construct a family $\mathscr{F}$ of real-valued functions defined on S and an operator $J : \mathscr{F} \to X$.

For a simple function $f = \sum_{i=1}^{n} c_i\chi_{E_i}$ (E_i disjoint) put $J_0(f) = \sum_{i=1}^{n} c_i\alpha(E_i) \in X$.

Evidently, $J_0(f) + J_0(g) = J_0(f + g)$ and $f \geqq 0$ implies $J_0(f) \geqq 0$. Moreover, we prove that $f_n \searrow 0$ implies $J_0(f_n) \searrow 0$.

Let δ be a positive real number, $G_n = \{x : f_n(x) \geqq \delta\}$, $M = \max f_1$. Then $G_n \supset G_{n+1}$ $(n = 1, 2, \ldots)$, $\bigcap_{n=1}^{\infty} G_n = \emptyset$, hence $\alpha(G_n) \searrow 0$. Further, we have

$$J_0(f_n) = J_0(f_n \chi_{G_n}) + J_0(f_n \chi_{S-G_n}) \leqq M\alpha(G_n) + \delta\alpha(S) .$$

Now according to Theorem 9 of [7] there is a set $\mathscr{F}$ including all simple functions and an extension J of J_0 satisfying conditions 2,3 and 4. J fulfills evidently also condision 1.

REFERENCES

[1] Duchoň M., *Прямое произведение скалярной и векторной мер*, Mat.-fyz. časop. *16* (1966), 274—281.

[2] Duchoň M., Kluvánek I., *Inductive tensor product of vector-valued measures*, Mat. časop. *17* (1967), 108—112.

[3] Duchoň M., *On the projective tensor product of vector-valued measures*, Mat. časop. *17* (1967), 113—120.

[4] Duchoň M., *On the projective tensor product of vector-valued measures II*, Mat. časop. *19* (1969), 228—234.

[5] Канторович Л. В., Вулих Б. З., Пинскер А. Т., *Функциональный анализ в полуупорядоченных пространствах*, Москва 1950.

[6] Вулих Б. З., *Введение в теорию полуупорядоченных пространств*, Москва 1961.

[7] Riečan B., *О продолжении операторов со значениями в линейних полуупорядоченных пространствах*, Čas. pěstov. mat. *93* (1968), 459—471.

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