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Matematický časopis, Vol. 21 (1971), No. 2, 160--166

Persistent URL: `http://dml.cz/dmlcz/126430`

ON THE SUMMABILITY OF SUBSEQUENCES

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The present paper deals with regular matrix summability methods and their relations to the sets of limit points of transformed subsequences.

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First we introduce some preliminary results. Let $\{s_n\}_{n=1}^{\infty}$ be an arbitrary sequence. With each subsequence $\{s_{n_l}\}_{l=1}^{\infty}$ we can associate the number $x_0 = \sum\limits_{l=1}^{\infty} 2^{n_l}$. Conversely, let $0 < x \leqslant 1$ and

$$x = 0 \,.\, \alpha_1\alpha_2\alpha_3 \ldots \tag{1}$$

be its dyadic expansion with infinitely many 1's. Let $\{m_i\}_{i=1}^{\infty}$ be the set of all indices in (1) such that $\alpha_{m_i} = 1$. Using $\{m_i\}$ we can associate to x the subsequence $\{s_{m_i}\}_{i=1}^{\infty}$, which will be denoted by $\{s(n, x)\}$. This one-to-one mapping of all subsequences of a sequence $\{s_n\}$ on the interval $(0,1\rangle$ have been utilized by Buck and Pollard [1] to study certain properties of subsequences.

Let $T = (a_{mn})$ be a regular summability method. Let (1) be the dyadic expansion of the number x with $\alpha_{n_k} = 1$ and $\alpha_n = 0$ for $n \neq n_k (k = 1, 2, 3, \ldots)$. Let us put

$$\sigma(m, x) = \sum_{k=1}^{\infty} a_{mk} s(k, x), \text{ where } s(k, x) = s_{n_k},$$

$$\varphi(k, m) = \sup \{\nu : \sum_{n=1}^{\nu} |a_{mn}| < k^{-1}\}$$

$$\psi(k, m) = \min \{\nu : \sum_{n=\nu+1}^{\infty} |a_{mn}| < k^{-1}\}$$

and $F(k) = \sup\limits_m \{|\varphi(k, m) - \psi(k, m)|\}$.

We now recall the definition of the homogeneous set and two sufficient conditions for the homogeneity of a set from [3]. Let $|A|$ ($|A|_e$) denote the Lebesque measure (the exterior Lebesque measure) of the set A.

Definition. *A set $M \subset (0, 1\rangle$ is said to be homogeneous if for two arbitrary intervals $I_1, I_2 \subset (0, 1\rangle$ the equality*

$$\frac{|I_1 \cap M|_e}{|I_1|} = \frac{|I_2 \cap M|_e}{|I_2|}$$

holds.

Theorem A. *A set $M \subset (0, 1\rangle$ is homogeneous if an arbitrary interval $I \subset$ $\subset (0, 1\rangle$ can be divided into a countable system of intervals I_n with the following properties:*

a) *every two intervals $I_{n_1} \neq I_{n_2}$ have at most one endpoint in common,*

b) $|\bigcup_{n=1}^{\infty} I_n| = |I|$

c) *for every n the set $I_n \cap M$ is geometrically analogous either to the set M or to a set M_n, being distinct from M in at most a set of the measure zero.*

Theorem B. *Let $M \subset (0, 1\rangle$ be such a measurable set that for an arbitrary irrational number $x_0 \in (0, 1\rangle$, $x_0 = 0 . \alpha_1\alpha_2\alpha_3 \ldots$ either all or none of numbers $x_p = 0 . \alpha_{p+1}\alpha_{p+2} \ldots$ $(p = 0,1, 2, \ldots)$ belong to M. Then M is a homogeneous set and $|M| = 0$ or 1.*

In [1] and [4] a restricted definition of the homogeneous set is used. This definition is convenient as a criterion and is given in Theorem C.

Theorem C. *Let a measurable set $M \subset (0, 1\rangle$ have the following property: If $x = 0 . \alpha_1\alpha_2\alpha_3 \ldots$ is the dyadic expansion of a point x of M, then the point obtained by altering a finite number of α_i also belongs to M. Then M is a homogenous set and $|M| = 0$ or 1.*

Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence and T be a regular matrix summability method. Golubov [2] (Theorem 2) proved that there is a set Q residual in $(0, 1\rangle$ such that for every $x \in Q$ the inclusion $\{\sigma(n, x)\}' \supset \{s_n\}'$ is valid. ($\{t_n\}'$ denotes the set of all limit points of the sequence $\{t_n\}_{n=1}^{\infty}$). The following asserts that the set

$$Q_1 = \{x \in (0, 1\rangle : \{\sigma(n, x)\}' \supset \{s_n\}'\}$$

is measurable. Inclusion $Q_1 \supset Q$ is evident.

Theorem 1. *Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence, T a regular matrix summability method. Then the set Q_1 is a union of a countable set and a G_δ set.*

Proof. Let M be the set of all rational numbers of the interval $(0, 1\rangle$.

Let $X = (0, 1\rangle - M$, $Q_2 = Q_1 \cap X$ and $\{u_1, u_2, u_3, \dots u_m, \dots\}$ $(m = 1, 2, \dots)$ be a dense subset of the set $\{s_n\}'$. Let us put

$$S^k_{mnp} = \{x \in X: |\, \sigma(n + p, x) - u_m| < k^{-1}\}$$

and

$$S^k_{mn} = \bigcup_{p=1}^{\infty} S^k_{mnp}.$$

We shall show that

$$Q_2 = \bigcap_m \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} S^k_{mn}\,. \tag{2}$$

Let x belong to the right-hand side of (2). Then the statement

$$\forall_m \forall_k \forall_n \exists_p\, |\sigma(n + p, x) - u_m| < k^{-1} \tag{3}$$

is valid. The validity of the statement $u_m \in \{\sigma(n, x)\}'$ for every m follows from (3). Since $\overline{\{u_m\}} = \{s_n\}$,' we obtain $\{s_n\}' \subset \{\sigma(n, x)\}'$. As $x \in X$, we have $x \in Q_2$.

Let $x \in Q_2$. Let m, n, k be three arbitrary positive integers. Since $\{\sigma(n, x)\}' \supset \supset \{s_n\}' \supset \{u_m\}$, we have $u_m \in \{\sigma(n, x)\}'$ for every m. Also there is a strongly increasing sequence of positive insegers $n_1 < n_2 < n_3 < \dots$ such that $\sigma(n_l, x) \to \to u_m$. We can choose l such that $n_l > n$ and $|\sigma(n_l, x) - u_m| < k^{-1}$. Let $p = n_l - n$. Then $\sigma(n_l, x) = \sigma(n + p, x)$ and $|\sigma(n + p, x) - u_m| < k^{-1}$. Since $x \in X$, from the definition of S^k_{mnp} it follows that $x \in S^k_{mn}$ for arbitrary m, n, k and thus x belongs to the right-hand side of (2).

We now show that S^k_{mnp} is an open set in X. It is suffitient to prove that $\sigma(n + p, x)$ is a continuos function of the variable $x \in X$. Let x_0 be an arbitrary point from X. Since $\{s_m\}_{m=1}^{\infty}$ is a bounded sequence, there exists a number $C > 0$ such that $|s_m| \leqslant C$ $(m = 1, 2, \dots)$. Let η be an arbitrary number. Let us choose N such that $\sum_{l=N+1}^{\infty} |a_{n+pl}| < \eta/2C$. Let $x_0 = 0\,.\,\alpha^0_1\alpha^0_2 \dots$ be the dyadic expansion of x_0 with infinitely many digits equal to 1. Let $N' \geqslant N$ be a positive integer such that among the first N' digits of the dyadic expansion of x_0 exactly N digits are equal to 1. Let O_{x_0} be an open set in X such that $x_0 \in O_{x_0}$ and for each $x \in O_{x_0}$, $x = 0\,.\,\alpha_1\alpha_2\alpha_3 \dots$ we have $\alpha_l = \alpha^0_l$ $(l = 1, 2, \dots N')$. Then $s(l, x) = s(l, x_0)$ for $x \in O_{x_0}$ and $l = 1, 2, \dots N$. Thus we get

$$|\sigma(n + p, x) - \sigma(n + p, x_0)| \leqslant \sum_{l=N+1}^{\infty} |a_{n+pl}|\, |s(l, x) - s(l, x_0)| \leqslant$$

$$\leqslant 2C \sum_{l=N+1}^{\infty} |a_{n+pl}| < \eta$$

Hence $\sigma(n+p, x)$ is a continuous function of the variable x and the sets S^k_{mnp} are open in X. Q_2 is a G_δ set in X, as it follows from (2) and Q_2 is a G_δ set in $(0, 1\rangle$, too. The set $M \cap Q_1$ is countable. $Q_1 = Q_2 \cup (M \cap Q_1)$ and the theorem is proved.

The following theorem gives a sufficient condition for Q_1 to be a homogeneous set.

Theorem 2. *Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence. Let $T = (a_{mn})$ be a regular matrix summability method satisfying the following two conditions*

(i) $\limsup\limits_{m\to\infty} \sup\limits_{n} |a_{mn}| = 0$

(ii) $F(k) = o(k)$.

Then Q_1 is a homogeneous set.

Corollary. *If a sequence $\{s_n\}_{n=1}^{\infty}$ and a regular summability method satisfy the conditions of Theorem 2, then $|Q_1| = 0$ or 1.*

To prove Theorem 2 we need the following lemma.

Lemma. *Let T be a regular summability method defined by the matrix (a_{mn}). Then there exists a positive integer m_0 such that $\varphi(k, m) < +\infty$ and $\varphi(k, m) \leqslant \leqslant \psi(k, m)$ for each $m \geqslant m_0$ and $k \geqslant 3$.*

Proof. Let $A_m = \sum_{n=1}^{\infty} a_{mn}$ and $0 < \eta < 2^{-1}$. It is known that $A_m \to 1$ and obviously $\sum_{n=1}^{\nu} |a_{mn}| \geqslant \sum_{n=1}^{\nu} a_{mn}$. Let us choose a natural number $m_0 = m_0(\eta)$ such that

(4) $|A_m{}^{-1}| < \eta/2$

for $m \geqslant m_0$. Since $\sum_{n=1}^{\nu} a_{mn} \to A_m$ for $\nu \to \infty$, then there exists a $\nu_0(m)$ such that

(5) $\quad |\sum_{n=1}^{\nu} a_{mn} - A_m| < \eta/2$

for $\nu > \nu_0(m)$. The inequality $\sum_{n=1}^{\nu} a_{mn} > 1 - \eta$ follows from (4) and (5). If we consider $k \geqslant 3$, then $1 - \eta > k^{-1}$. Thus $\varphi(k, m) < \infty$ for $k \geqslant 3$ and $m \geqslant m_0$. Suppose that $\varphi(k, m) > \psi(k, m)$ for some $k \geqslant 3$ and $m \geqslant m_0$. Then

$$\sum_{n=1}^{\varphi(k,m)} |a_{mn}| + \sum_{n=\psi(k,m)+1}^{\infty} |a_{mn}| \geqslant \sum_{n=1}^{\infty} |a_{mn}| \geqslant A_m$$

Since $\sum_{n=1}^{\varphi(k,m)} |a_{mn}| < k^{-1}$ and $\sum_{\psi(k,m)+1}^{\infty} |a_{mn}| < k^{-1}$,

we have by (4)

$$\frac{2}{3} \geqslant \frac{2}{k} \geqslant A_m > 1 - \frac{\eta}{2} > \frac{3}{4}.$$

This contradiction completes the proof of the lemma.

Proof of Theorem 2. The validity of the statement "Q_1 is a measurable set" can be easily verified by Theorem 1. We are permitted to consider only irrational numbers of Q_1 when investigating the homogeneity of the set Q_1 by Theorem A. x_p has the same meaning as in Theorem B. Let $|s_n| \leqslant C$ $(n = 1, 2, \ldots)$ and $\varepsilon > 0$. Let us choose $k_0 \geqslant 3$ and M_0 such that $\varphi(k, m) < \infty$, $\varphi(k, m) \leqslant \psi(k, m)$, $4C/k < \varepsilon/2$, $2CF(k)/k < \varepsilon\backslash 2$ for $k > k_0$ and $m > M_0$. It can be done according to (ii). Let us choose a fixed $k > k_0$. We conclude from (i) that there exists an $M_1 \geqslant M_0$ such that $|a_{mn}| < k^{-1}$ $(n = 1, 2, 3, \ldots)$ $m > M_1$. If $x_0 \in Q_1$ and k, $m > M_1$ are choosen in the above mentioned way, we obtain

$$|\sigma(m, x_0) - \sigma(m, x_p)| = |\sum_{\nu=1}^{\infty} a_{m\nu} s_{n_\nu} - \sum_{\nu=1}^{\infty} a_{m\nu} s_{p_\nu}| \leqslant \sum_{\nu=1}^{\varphi(m,k)} |a_{m\nu}|\, |s_{n_\nu} - s_{p_\nu}| +$$

$$+ \sum_{\varphi(k,m)+1}^{\psi(k,m)} |a_{m\nu}|\, |s_{n_\nu} - s_{p_\nu}| + \sum_{\psi(k,m)+1}^{\infty} |a_{m\nu}|\, |s_{n_\nu} - s_{p_\nu}| \leqslant \frac{2C}{k} +$$

$$+ \frac{2C\,|\psi(k,m) - \varphi(k,m)|}{k} + \frac{2C}{k} < \varepsilon.$$

Hence $\lim_{m\to\infty} |\sigma(m, x_0) - \sigma(m, x_p)| = 0$ holds for every p. Thus $\{\sigma(m, x_p)\}' \supset \supset \{\sigma(m, x_0)\}' \supset \{s_n\}'$ and $x_p \in Q_1$. Q_1 is a homogeneous set according to Theorem B.

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Henceforth a subsequence of the sequence $\{s_n\}_{n=1}^{\infty}$ means the sequence $\{\alpha_n s_n\}_{n=1}^{\infty}$, where $\alpha_n = 0$ or 1 and $\alpha_n = 1$ for infinitely many n. Also a one-to-one mapping betwen the set of all subsequences of the sequence $\{s_n\}_{n=1}^{\infty}$ and the inserval $(0, 1\rangle$ can be defined in an analogous way as in the first part of the paper. Let $x \in (0, 1\rangle$ and $x = 0 . \alpha_1\alpha_2\alpha_3 \ldots$ be the dyadic expansion with infinitely many digits equal to 1. If $T = (a_{mn})$ is a regular matrix summability method, then we put

$$\tau(m, x) = \sum_{n=1}^{\infty} a_{mn}\alpha_n s_n.$$

An analogous theorem to Theorem 1 can be obtained if $\sigma(n, x)$ is replaced by $\tau(n, x)$. The following theorem is analogous to Theorem 2 but any other conditions for the summability method except that of regularity are not required.

Theorem 3. *Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence and T a regular summability method. Then*

$$Q_1 = \{x \in (0, 1\rangle : \{\tau(n, x)\}' \supset \{s_n\}'\}$$

is a measurable homogeneous set (and hence $|Q_1| = 0$ or 1).

Proof. It can be easily shown that Q_1 is a measurable set. Let $x = 0 . \alpha_1\alpha_2\alpha_3 \ldots$ ($\alpha_i = 0$ or 1 and for infinitely many i we have $\alpha_i = 1$) belong to Q_1. Let $y = 0 . \beta_1\beta_2\beta_3 \ldots$ be a point obtained by altering a finite number of the α_i and $|s_n| \leqslant C$, $C > 0$ ($n = 1, 2, 3, \ldots$). Let $\varepsilon > 0$ and

$$K_0 = \min\{k : \alpha_i = \beta_i \text{ for } i > k\}.$$

Since T is a regular matrix summability method, there exists an integer M_0 such that $|a_{mn}| < \varepsilon/K_0C$ for $m \geqslant M_0$ and $n = 1, 2, \ldots, K_0$. Then we have for $m \geqslant M_0$

$$|\tau(m, x) - \tau(m, y)| = |\sum_{n=1}^{\infty} a_{mn}\alpha_n s_n - \sum_{n=1}^{\infty} a_{mn}\beta_n s_n| \leqslant \sum_{n=1}^{\infty} |a_{mn}|\,|\alpha_n - \beta_n|\,|s_n| < \varepsilon .$$

Thus we obtain that $\lim_{m\to\infty} |\tau(m, x) - \tau(m, y)| = 0$ and therefore $y \in Q_1$. Q_1 is a homogeneous set according to Theorem C.

Remark. The assumption of regularity in Theorem 3 is essential. If the regularity of a summability method is not required, then there exists a summability method summing every convergent sequence and a bounded sequence such that $|Q_1| = 2^{-p}$. The construction of this method for $p = 1$ is given in the following example.

Example. Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence for which $s_1 \neq 0$, $s_n \to \alpha$ and $|s_n| \geqslant C$. We define the summability method A by the matrix (a_{mn}), where $a_{m1} = 3Cs_1^{-1}$ ($m = 1, 2, \ldots$), $a_{mn} = 1$ ($m = 2, 3, \ldots$) and $a_{nk} = 0$ for $k \neq 1$, n. Let $\tau(n, x) = \sum_{k=1}^{\infty} a_{nk}\alpha_k s_k$. Then $\tau(n, x) = \alpha_n s_n$ for $x \in (0, \frac{1}{2}\rangle$ and $\tau(n, x) = 3C + \alpha_n s_n$ for $x \in (\frac{1}{2}, 1\rangle$. It follows from the above that

$$|\{x \in (0, 1\rangle : \{\tau(n, x)\}' \supset \{s_n\}'\}| = \tfrac{1}{2}.$$

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Received October 17, 1969.

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