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GENERALIZED GROUPS WITH THE WELL-ORDERED SET OF IDEMPOTENTS

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Dedicated to Professor Štefan Schwarz on the occasion of his fiftieth birthday

A semigroup G is called a *generalized group* (or an *inverse semigroup*) if for any $g \in G$ there exists a unique element $g^{-1} \in G$ (the generalized inverse of g) such that

$$gg^{-1}g = g, \quad g^{-1}gg^{-1} = g^{-1}$$

holds.

Generalized groups have been introduced by V. V. Wagner (В. В. Вагнер) [1] in 1952. The definition given above is due to A. E. Lieber (А. Е. Либбер) [2]. In the following we suppose the basic properties of generalized groups to be known (see, e.g. [3], [4]).

A. H. Clifford considered in 1941 [6] a special class of generalized groups in which for any $g \in G$

$$gg^{-1} = g^{-1}g$$

holds. We shall call this class of generalized groups "*Cliffordian generalized groups*".

In every generalized group we can introduce a binary relation ω by:

$$(g_1, g_2) \in \omega \leftrightarrow g_1g_1^{-1} = g_1g_2^{-1}.$$

(See also [1], [5] for other definitions of ω .) ω is known to be an order relation.

We use the notation $g_1 < g_2$ as synonymous with $(g_1, g_2) \in \omega$.

ω is stable and involutorily invariant, i.e.

$$g_1 < g_2, \quad g_3 < g_4 \rightarrow g_1g_3 < g_2g_4, \\ g_1 < g_2 \rightarrow g_1^{-1} < g_2^{-1}.$$

We shall call ω the *canonical order relation* of G .

Denote by I the set of all idempotents of G . It is known (and easy to prove) that I is a commutative subsemigroup of G and for every couple $i_1, i_2 \in I$ we have

$$i_1 < i_2 \leftrightarrow i_1i_2 = i_1. \tag{1}$$

The purpose of this paper is to prove the following

Theorem. *Let G be a generalized group. If the set I of all idempotents $\in G$ is well-ordered (in the canonical order relation), then G is a Cliffordian generalized group.*

Proof. Let I be well-ordered (in ω), g any element $\in G$ and $[g]$ the generalized subgroup generated by g . Clearly $[g] \subset G$.

Note first: If $gg^{-1} \neq g^{-1}g$ then (since I is linearly ordered and $gg^{-1}, g^{-1}g \in I$) we have either $gg^{-1} < g^{-1}g$ or $g^{-1}g < gg^{-1}$.

In the following we shall suppose $gg^{-1} < g^{-1}g$. (The second case can be treated similarly.) The relation (1) implies $g^{-1}ggg^{-1} = gg^{-1}$ and multiplying by g to the right we have $g^{-1}g(gg^{-1}g) = gg^{-1}g$, i.e.

$$g^{-1}g^2 = g. \tag{2}$$

It is easy to see (cf. [7], n. 1, 3, 1) that $[g]$ coincides with the subsemigroup of G generated by g and g^{-1} and every $x \in [g]$ can be written in the form

$$x = (g^{-1})^k g^l (g^{-1})^m \quad \text{with } l \geq k \geq 0, \quad l \geq m \geq 0, \quad l > 0 \tag{3}$$

([7], Theorem 1,3). It should be noted that the representation of x in this form is not uniquely determined.

Lemma 1. *Under the suppositions mentioned above every $x \in [g]$ can be written in one of the following forms:*

- a) either $x = g^l \cdot (g^{-1})^m$ with $l > 0, \quad m \geq 0,$
- b) or $x = (g^{-1})^m$ with $m > 0,$
- c) or $x = (g^{-1})^k \cdot g$ with $k > 0.$

To prove this we proceed as follows. Let l be the least positive integer such that $x = (g^{-1})^k g^l (g^{-1})^m$. If $k = 0$, we have the form a) and there is nothing more to prove. Suppose therefore $k > 0$

- a) If $l > 1$ and $m \geq 0$, we have

$$\begin{aligned} x &= (g^{-1})^k g^l (g^{-1})^m = (g^{-1})^{k-1} g^{-1} g^2 \cdot g^{l-2} (g^{-1})^m = \\ &= (g^{-1})^{k-1} \cdot g \cdot g^{l-2} (g^{-1})^m = (g^{-1})^{k-1} g^{l-1} (g^{-1})^m, \end{aligned}$$

contrary to the choice of l . Hence for $k > 0$ the number l cannot be > 1

- b) Let $l = 1$ and $m \geq 1$.

Then

$$x = (g^{-1})^{k-1} g^{-1} g g^{-1} (g^{-1})^{m-1} = (g^{-1})^{k-1} \cdot g^{-1} \cdot (g^{-1})^{m-1} = (g^{-1})^{k+m-1},$$

hence x is of the form b).

- c) Let $l = 1$ and $m = 0$. Then $x = (g^{-1})^k g$ is of the form c).

Lemma 2. Any idempotent $x \in [g]$ is of the form

A. either $x = g^{-1}g$,

B. or $x = g^l(g^{-1})^l$, $l \geq 1$.

Before proving this recall that if $x = (g^{-1})^k g^l (g^{-1})^m$, then $x^{-1} = g^m \cdot (g^{-1})^l g^k$.
Note further: If $m \geq 2$ then

$$(g^{-1})^m \cdot g^m = (g^{-1})^{m-1} \cdot g^{-1} \cdot g^2 g^{m-2} = (g^{-1})^{m-1} \cdot g^{m-1} = \dots = g^{-1} \cdot g.$$

Let now be x an idempotent $\in [g]$. Then $x = xx^{-1}$.

In the case a) we have $x = g^l (g^{-1})^m g^m (g^{-1})^l$. If $m = 0$, $x = g^l \cdot (g^{-1})^l$. If $m = 1$, $x = g^l \cdot g^{-1} g (g^{-1})^l = g^l (g^{-1} g g^{-1}) (g^{-1})^{l-1} = g^l g^{-1} (g^{-1})^{l-1} = g^l \cdot (g^{-1})^l$. If $m \geq 2$, $x = g^l \cdot g^{-1} \cdot g (g^{-1})^l = g^l (g^{-1})^l$.

In the case b) we have $x = xx^{-1} = (g^{-1})^m g^m = g^{-1}g$.

In the case c) we have $x = (g^{-1})^k g \cdot g^{-1} g^k = (g^{-1})^k (g g^{-1} g) g^{k-1} = (g^{-1})^k g g^{k-1} = (g^{-1})^k \cdot g^k = g^{-1}g$.

This proves our lemma.

Now it can be easily verified that

$$\dots g^{l+1} \cdot (g^{-1})^{l+1} < g^l \cdot (g^{-1})^l < \dots < g g^{-1} < g^{-1}g,$$

i.e. the idempotents $\in [g]$ form a decreasing chain. The well-ordering of I implies that this chain breaks up, i.e. $[g]$ possesses a finite number of idempotents. Now $g^{-1}g$ being the greatest idempotent $\in [g]$ is an identity of $[g]$. It follows $g g^{-1} = g^{-1}g$. ([2], Theorem 8.) Thus G is Cliffordian.

Corollary 1. If the set of all idempotents of a generalized group is finite and linearly ordered, then the generalized group is Cliffordian.

Corollary 2. ([2], Theorem 9.) Every generalized group having only two idempotents is Cliffordian.

We conclude with the following conjecture (the converse of our Theorem) which we are not able to prove or disprove:

Let I be an idempotent generalized group having the property: If I is isomorphic to the generalized group of all idempotents of a generalized group G , then G is Cliffordian. Then the canonical order of I is the well-order.

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ОБОБЩЕННЫЕ ГРУППЫ С ВПОЛНЕ УПОРЯДОЧЕННЫМИ МНОЖЕСТВАМИ ИДЕМПОТЕНТОВ

Борис Моисеевич Шайн

Резюме

Доказана теорема: Если множество всех идемпотентов обобщенной группы вполне упорядочено каноническим отношением порядка этой обобщенной группы, то данная обобщенная группа является клиффордовой (т. е. удовлетворяет условию $gg^{-1} = g^{-1}g$ для любого элемента g).