Sidney A. Morris; N. Kelly

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VARIETIES OF TOPOLOGICAL GROUPS GENERATED BY GROUPS WITH INVARIANT COMPACT NEIGHBOURHOODS OF THE IDENTITY

SIDNEY A. MORRIS—N. KELLY*

1. **Introduction.** In his book "L'integration dans les groupes topologiques" Weil asserted that if some compact neighbourhood of the identity of a topological group is invariant under all inner automorphisms then there are arbitrarily small neighbourhoods which are invariant under all inner automorphisms; that is, any IN-group is a SIN-group. Mostow [11] showed that this assertion is false. More recently the work of Grosser and Moskowitz [2, 3] and Hofmann and Mostert [5] has clarified the relationships of the various interesting compactness conditions in topological groups. (Further information on IN-groups appears in Poguntke [13] and Ordman and Morris [12].)

In [7] we investigated varieties of topological groups generated by SIN-groups and maximally almost periodic groups. (For a discussion of varieties of topological groups and a list of references, see Morris [6]) Our aim here is to examine varieties generated by IN-groups. Since IN-groups are locally compact and one of our varietal operations is the forming of infinite cartesian products we cannot expect that every group in a variety generated by IN-groups is an IN-group. However, it is reasonable to hope that every locally compact group in a variety generated by IN-groups is an IN-group. (For results of this type see [1, 8, 9, 10].) We have only been able to prove this when we also assume some connectedness condition.

2. **Preliminaries.** A non-empty class **V** of topological groups (not necessarily Hausdorff) is said to be a *variety* of it is closed under the operations of taking subgroups, quotient groups, arbitrary cartesian products and isomorphic images. The smallest variety containing a class Ω of topological groups is said to be the *variety generated by* Ω and is denoted by $V(\Omega)$.

If Ω is any class of topological groups, then $S(\Omega)$ denotes the class of all

* This paper was written while the first author was a United Kingdom Science Research Council Senior Visiting Fellow at the University of Wales.

topological groups isomorphic to subgroups of members of Ω. Similarly we define the operators $\bar{S}$, $\bar{Q}$, C and D where they denote closed subgroup, separated quotient, arbitrary cartesian product and finite product, respectively.

Theorem [I]. *If Ω is a class of topological groups and G is a Hausdorff group in $V(\Omega)$, then $G \in SC\bar{Q}\bar{S}D(\Omega)$.*

A topological group G is said to be an IN-group if there exists a compact neighbourhood of the identity in G which is invariant under all inner automorphisms of G.

3. **Results. Lemma.** *Let Ω be a class of topological groups each of which has the property that the closure of its commutator subgroup is compact. Then every complete Hausforff group G in $V(\Omega)$ has this property.*

Proof. By the theorem in Section 2, $G \in SC\bar{Q}\bar{S}D(\Omega)$. In fact, since G is complete, $G \in \bar{S}C\bar{Q}\bar{S}D(\Omega)$. It is a routine matter to verify that the property referred to in the statement of the Lemma is preserved by each of the operations $\bar{Q}$, $\bar{S}$, C and D. Thus G has the required property.

To see the relevance of the above Lemma we state two results:

(A) [2, Table IV]. If G is a locally compact group with the closure of its commutator subgroup compact, then G is an IN-group.

(B) [2, Table III]. If G is a connected IN-group, then the closure of its commutator subgroup is compact.

With these results in hand we can now prove:

Theorem 1. *Let Ω be a class of locally compact groups. If the component of the identity of each group in Ω is an* IN*-group, then every connected locally compact group G in $V(\Omega)$ is an* IN*-group.*

Proof. For any group H, let $K(H)$ denote the component of the identity of H and let H' denote the closure of its commutator subgroup. Then if $G_1, \ldots, G_n$ are members of Ω, we have $K(G_i)$ is an IN-group, for $i = 1, \ldots, n$. By (B) above, this implies that each $K(G_i)'$ is compact. Noting that

$$K(G_1 \times G_2 \times \ldots \times G_n) = K(G_1) \times \ldots \times K(G_n)$$

and hence that

$$K'(G_1 \times G_2 \times \ldots \times G_n) \subseteq K(G_1)' \times \ldots \times K(G_n)'$$

we see that for each $H \in D(\Omega)$, $K(H)'$ is compact. Similarly, we see that if $H \in \bar{S}D(\Omega)$ then $K(H)'$ is compact.

Now assume that H is such that $K(H)'$ is compact and let A be any separated quotient of H. Let $f\colon H \to A$ be the quotient homomorphism. By [4, Theorem 7.12] we see that $f(K(H))$ is dense in $K(A)$. Therefore, $f(K(H)')$ is dense in $K(A)'$. Since $K(H)'$ is compact, this implies $f(K(H)') = K(A)'$.

Consequently every group H in $\bar{Q}\bar{S}D(\Omega)$ has the property that $K(H)'$ is compact. Indeed we see that every group H in $\bar{S}C\bar{Q}\bar{S}D(\Omega)$ has $K(H)'$ compact.

Now $G \in V(\Omega)$, so by the theorem in Section 2, $G \in SC\bar{Q}\bar{S}D(\Omega)$. Since G is locally compact it is complete and thus $G \in \bar{S}C\bar{Q}\bar{S}D(\Omega)$. Then, by our above remarks, $K(G)'$ is compact. Since G is connected this says G' is compact. Finally, by (A) above, we have that G is an IN-group.

Corollary. *Let Ω be a class of* IN-*groups. Then any connected locally compact group in $V(\Omega)$ is an* IN-*group.*

Proof. This follows immediately from Theorem 1 by noting that if H is an IN-group then $K(H)$ is an IN-group.

Theorem 2. *Let Ω be a class of connected* IN-*groups. Then any locally compact group G in $V(\Omega)$ is an* IN-*group.*

Proof. By (B) above, every member of Ω has the closure of its commutator subgroup compact. By the Lemma this implies that every complete topological group in $V(\Omega)$ has the closure of its commutator subgroup compact. In particular, since G is complete it also has this property. Then by (A) above, G is an IN-group.

We conclude with a question:

Question. Is every locally compact group in a variety generated by IN-groups necessarily an IN-group?

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Dr. S. A. Morris,
University College of North wales,
Bangor, Wales, LL57 2UW
U.K.

Professor N. Kelly,
University of Glenrowan,
Glenrowan, Victoria,
AUSTRALIA