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AN EXTENSION OF THE DANIELL INTEGRATION SCHEME

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The well known Daniell integration scheme can be applied simultaneously to a measure and an integral. An integral and a measure can namely be regarded as functions on partially ordered sets: the integral on a set of functions, the measure on a set of sets. An extension theory for functions on partially ordered sets can be constructed in such a way that integration and measure extension theories are its special cases.

It is known that the measure extension theory can be deduced from the Daniell integration theory, but from the general extension theory we can get not only the classical two cases. We can obtain also a measure extension theory for Boolean σ-algebras (and sometimes even for certain classes of non-distributive lattices) and on the other hand a Daniell integration theory for σ-complete lattice ordered groups. Hence the mentioned method is very economical. These facts are well known (see e. g. [1], [2], [3]).

In the paper we present an improvement of the mentioned results. We want to get not only a theorem on the extension of measures, but also a theorem on the extension of so-called subadditive measures (i. e. subadditive and upper and lower semicontinuous set functions). Of course, simultaneously we get a theory of subadditive integrals.

In comparing with the preceding papers ([2], [3]) we assume less about the original function and we prove less about the extension. In the theory presented here we need a more elaborate structure on the domain of the studied functions, but the proofs are more simple.

In what follows we denote by $\cup$, $\cap$ or $\bigcup$, $\bigcap$, resp. the lattice operations. If $a_n \leqq a_{n+1}$ $(n = 1, 2, \ldots)$ and $a = \bigcup_{n=1}^{\infty} a_n$ is the least upper bound of $\{a_n\}_{n=1}^{\infty}$, then we write $a_n \nearrow a$. The symbol $a_n \searrow a$ has an analogous meaning.

1. General theory

We start with a conditionally σ-complete lattice S with two further binary operations $+$, $-$ and we list axioms. (If S is a lattice of real-valued functions, then $+$, $-$ are the usual sum and difference of the functions; if S is a lattice of sets, then $+$, $-$ are the set theoretical union and difference.)

1.1. $+$ is associative and commutative.

1.2. There is an element $0 \in S$ such that $x - x = 0$ for every $x \in S$.

1.3. If $a \leqq x \leqq b$ and $c \leqq y \leqq d$, then
$a - d \leqq x - y \leqq b - c$, $a + c \leqq x + y \leqq b + d$,
$(b - c) - (a - d) \leqq (b - a) + (d - c)$,
$(b + d) - (a + c) \leqq (b - a) + (d - c)$,
$b - a \leqq (b - x) + (x - a)$.

1.4. If $a \geqq c$, $b \geqq d$, then
$a \cup b - c \cup d \leqq (a - c) + (b - d)$,
$a \cap b - c \cap d \leqq (a - c) + (b - d)$.

1.5. If $a_n \nearrow a$, $b_n \nearrow b$, then
$a_n + b_n \nearrow a + b$, $a_n \cap b_n \nearrow a \cap b$.

1.6. If $a_n \searrow a$, $b_n \searrow b$, then
$a_n + b_n \searrow a + b$, $a_n \cup b_n \searrow a \cup b$.

1.7. If $a_n \nearrow a$, $b_n \searrow b$, then
$a_n - b_n \nearrow a - b$, $b_n - a_n \searrow b - a$.

Now let A be a sublattice of S closed under the operations $+$ and $-$. We also assume that to any $x \in S$ there are $a_n, b_n \in A$ such that $\bigcup_{n=1}^{\infty} b_n \in S$, $\bigcap_{n=1}^{\infty} a_n \in S$ and $\bigcap_{n=1}^{\infty} a_n \leqq x \leqq \bigcup_{n=1}^{\infty} b_n$.

Further let $J_0 : A \to R$ be a function satisfying the following axioms.

(i) $J_0(a + b) \leqq J_0(a) + J_0(b)$ for all $a, b \in A$.

(ii) $J_0(b) \leqq J_0(b - a) + J_0(a)$ for all $a, b \in A$.

(iii) $a \leqq b \Rightarrow J_0(a) \leqq J_0(b)$ for all $a, b \in A$.

(iv) If $a_n \searrow 0$, $a_n \in A$ $(n = 1, 2, \ldots)$, then $\lim\limits_{n\to\infty} J_0(a_n) = 0$.

(v) If $a_n \leqq a_{n+1} \leqq x \in S$, $a_n \in A$ $(n = 1, 2, \ldots)$ and $\{J_0(a_n)\}_{n=1}^{\infty}$ is bounded, then $\lim\limits_{n\to\infty} J_0(a_{n+1} - a_n) = 0$.

From (iv), 1.2, 1.7 and (ii) we get

$$a_n \nearrow a,\ a_n, a \in A\ (n = 1, 2, \ldots) \Rightarrow J_0(a_n) \nearrow J_0(a),$$

$$b_n \searrow b,\ b_n, b \in A\ (n = 1, 2, \ldots) \Rightarrow J_0(b_n) \searrow J_0(b).$$

The further extension process is described in the following definition. (Of course, we shall have to prove that the definitions of J^+, J^- and J are correct.)

Definition. *We put* $B = \{b \in S; \exists\, a_n \in A,\ a_n \nearrow b\}$, $C = \{c \in S; \exists\, a_n \in A,\ a_n \searrow c\}$. *Further we define* $J^+ : B \to R \cup \{\infty\}$, $J^- : C \to R \cup \{-\infty\}$ *by the formulas*

$$J^+(b) = \lim_{n\to\infty} J_0(a_n), \quad \textit{where} \quad a_n \nearrow b,$$

$$J^-(c) = \lim_{n\to\infty} J_0(a_n), \quad \textit{where} \quad a_n \searrow c.$$

Finally we put

$$L = \{x \in S;\ \forall\, \varepsilon > 0\ \exists\, b \in B,\ c \in C,\ c \leqq x \leqq b,\ J^+(b - c) < \varepsilon\}$$

and we define $J : L \to R$ *by the formula*

$$J(x) = \inf \{J^+(b);\ b \geqq x,\ b \in B\}.$$

Lemma 1. *If* $a_n \nearrow a$, $b_n \nearrow b$, $a_n, b_n \in A$ $(n = 1, 2, \ldots)$, $a \leqq b$, *then*

$$\lim_{n\to\infty} J_0(a_n) \leqq \lim_{n\to\infty} J_0(b_n).$$

Proof. According to 1.5 and 1.7 we have

$$a_m \cap b_n \nearrow a_m \quad (n \to \infty)$$

and

$$a_m - a_m \cap b_n \searrow 0 \quad (n \to \infty).$$

Hence according to (ii) and (iii) we obtain

$$0 \leqq J_0(a_m) - J_0(a_m \cap b_n) \leqq J_0(a_m - a_m \cap b_n)$$

and with respect to (iv)

$$\lim_{n\to\infty} (J_0(a_m) - J_0(a_m \cap b_n)) = 0.$$

Therefore

$$J_0(a_m) = \lim_{n\to\infty} J_0(a_m \cap b_n) \leqq \lim_{n\to\infty} J_0(b_n).$$

Evidently, an analogous assertion to Lemma 1 holds also for the set C. These assertions show that the definitions of J^+ and J^- are correct.

Proposition 1. *B and C are sublattices of S closed under the operation $+$. The functions J^+, J^- are increasing. For any $b_1, b_2 \in B$ and $c_1, c_2 \in C$ the following inequalities hold*

$$J^+(b_1 + b_2) \leqq J^+(b_1) + J^+(b_2),\ J^-(c_1 + c_2) \leqq J^-(c_1) + J^-(c_2).$$

If $b \in B$, $c \in C$, then $b - c \in B$, $c - b \in C$ and

$$J^+(b) \leqq J^+(b - c) + J^-(c),\ J^-(c) \leqq J^-(c - b) + J^+(b).$$

Finally, if $b_n \in B$, $b_n \nearrow b$, $c_n \in C$, $c_n \searrow c$, then $b \in B$, $c \in C$ and there are $k_n \in A$, $m_n \in A$ such that $k_n \leqq b_n$, $m_n \geqq c_n$ and $k_n \nearrow b$, $m_n \searrow c$.

Proof. We prove only the last assertion for B, the other assertions being easy. By the definition of B there are $a_n^m \in A$ such that $a_n^m \nearrow b_n$ $(m \to \infty)$. Put $k_n = \bigcup_{i=1}^{n} a_i^n$. The sequence $\{k_n\}_{n=1}^{\infty}$ has all the desired properties. The sequence $\{m_n\}_{n=1}^{\infty}$ can be constructed similarly.

Proposition 2. *L is a sublattice of S closed under the operations $+$ and $-$. J is finite on L and $J(x) = \sup\{J^-(c);\ c \leqq x, c \in C\}$.*

Proof. The first assertion follows from 1.3 and 1.4. For proving the second assertion take $b \in B$, $c \in C$ such that $c \leqq x \leqq b$ and $J^+(b - c) < \varepsilon$. Then

$$J(x) \leqq J^+(b) \leqq J^+(b - c) + J^-(c) < \varepsilon + J^-(c) < \infty\,.$$

Similarly $J(x) > -\infty$, since

$$J(x) \geqq J^-(c) \geqq J^+(b) - J^+(b - c) > J^+(b) - \varepsilon > -\infty\,.$$

Further

$$J(x) \leqq J^+(b) \leqq J^+(b - c) + J^-(c) \leqq$$
$$\leqq \varepsilon + \sup\{J^-(c);\ x \geqq c \in C\}\,,$$

hence

$$J(x) \leqq \sup\{J^-(c);\ x \geqq c \in C\}.$$

On the other hand

$$J^-(c) \leqq J^-(c - b) + J^+(b)$$

for arbitrary $c \in C$, $b \in B$ such that $c \leqq x \leqq b$. Since $c \leqq b$, i. e. $c \leqq c \leqq b$, $b \leqq b \leqq b$, we have according to 1.3 $c - b \leqq c - b \leqq b - b = O$, hence according to Lemma 1 and (iv)

$$J^-(c - b) \leqq J^-(b - b) = J_0(O) = 0\,.$$

Therefore

$$J^-(c) \leqq J^+(b),\ J^-(c) \leqq \inf\{J^+(b);\ x \leqq b \in B\} = J(x)$$

and hence

$$\sup\{J^-(c);\ x \geqq c \in C\} \leqq J(x).$$

Lemma 2. *If* $k_n \in A$, $k_n \nearrow b$ *and* $\{J_0(k_n)\}_{n=1}^{\infty}$ *is bounded, then* $\lim_{n\to\infty} J^+(b - k_n) = 0$.

Proof. Indirectly, let there exist such $\varepsilon > 0$ that for any n

$$J^+(b - k_n) > \varepsilon .$$

Since $k_m - k_1 \nearrow b - k_1$, there exists by Definition an $m_2 > 1$ such that

$$J_0(k_{m_2} - k_1) > \varepsilon .$$

Similarly there exists an $m_3 > m_2$ such that

$$J_0(k_{m_3} - k_{m_2}) > \varepsilon .$$

Continuing the process we obtain a sequence $k_1 \leqq k_{m_2} \leqq k_{m_3} \leqq \ldots$ such that $\{J_0(k_{m_n})\}_{n=1}^{\infty}$ is bounded, but

$$\lim_{n\to\infty} J_0(k_{m_n} - k_{m_{n-1}}) \neq 0$$

which is a contradiction to (v).

Proposition 3. *Let* $x_n \in L$ $(n = 1, 2, \ldots)$, $x_n \nearrow x$ *and* $\{J(x_n)\}_{n=1}^{\infty}$ *be bounded. Then* $x \in L$ *and* $J(x) = \lim_{n\to\infty} J(x_n)$.

Proof. Let ε be an arbitrary positive number. Take $a_n \in A$ such that $x \leqq \bigcup_{n=1}^{\infty} a_n = d$. Take $b_n \in B$, $c_n \in C$ such that $b_n \geqq x_n \geqq c_n$ and $J(b_n - c_n) <$ $< \varepsilon/2^n$. We can assume that $b_n \leqq d$. Further put $p_n = \bigcup_{i=1}^{n} b_i \in B$, $q_n = \bigcup_{i=1}^{n} c_i \in C$.

Then $q_n \leqq \bigcup_{i=1}^{n} x_i = x_n \leqq p_n$ and

$$J^+(p_n - q_n) = J^+(\bigcup_{i=1}^{n} b_i - \bigcup_{i=1}^{n} c_i) \leqq$$

$$\leqq J^+((b_1 - c_1) + (b_2 - c_2) + \ldots + (b_n - c_n)) \leqq$$

$$\leqq \sum_{i=1}^{n} J^+(b_i - c_i) < \varepsilon .$$

Since $b_n \leqq d$ $(n = 1, 2, \ldots)$, there exists $\bigcup_{n=1}^{\infty} b_n = b$; moreover $p_n \nearrow b$. According to Proposition 1 there are $k_n \in A$ such that $k_n \leqq p_n$, $k_n \nearrow b$. We get $b - k_n \searrow O$, $b - k_n \in B$. Further

$$J^+(b) = \lim_{n\to\infty} J_0(k_n) \leqq \lim_{n\to\infty} J^+(p_n) \leqq$$
$$\leqq \sup_n J^+(p_n - q_n) + \sup_n J^-(q_n) \leqq$$
$$\leqq \varepsilon + \sup_n J^-(q_n) \leqq \varepsilon + \sup_n J(x_n) < \infty .$$

Therefore according to Lemma 2

$$\lim_{n\to\infty} J^+(b - k_n) = 0 .$$

Further

$$J^+(b - q_n) \leqq J^+((b - p_n) + (p_n - q_n)) \leqq$$
$$\leqq J^+(b - p_n) + J^+(p_n - q_n) =$$
$$= J^+(b - k_n) + J^+(p_n - q_n) < 2\varepsilon$$

for sufficiently large n. Since $b \in B$, $q_n \in C$, $q_n \leqq x \leqq b$ and ε was an arbitrary positive number, $x \in L$.

Evidently $J(x_n) \leqq J(x)$ for every n, hence $\lim_{n\to\infty} J(x_n) \leqq J(x)$. On the other hand

$$J(x) \leqq J^+(b) \leqq J^+(b - q_n) + J^-(q_n) \leqq J^+(b - k_n) + J(x_n)$$

from which

$$J(x) = \lim_{n\to\infty} J(x_n) .$$

Note that also the dual assertion to Proposition 3 holds.

Theorem. *There is a lattice $L \subset S$ containing A and closed under the operations $+$, $-$ and there is a function $J: L \to R$, that is an extension of J_0 such that J satisfies the properties* (i), (ii), (iv′) *and* (iii). *If $x_n \in L$, $x \in S$, $x_n \nearrow x$ $(x_n \searrow x)$ and $\{J(x_n)\}_{n=1}^{\infty}$ is bounded, then $x \in L$ and $J(x) = \lim_{n\to\infty} J(x_n)$.*
If $I: L \to R$ is an extension of J_0 satisfying (iii) *and* (iv′), *then $I = J$.*

Proof. It suffices to prove the uniqueness. Put

$$N = \{x \in L;\ J(x) = I(x)\} .$$

By the assumption $N \supset A$. Moreover

$$N \supset \{b \in B;\ J^+(b) < \infty\}.$$

Indeed, take $a_n \in A$ such that $a_n \nearrow b$. Then $J^+(b) = \lim_{n\to\infty} J(a_n) = \lim_{n\to\infty} I(a_n) < < \infty$. Hence $I(b) = \lim_{n\to\infty} I(a_n) = \lim_{n\to\infty} J(a_n) = J^+(b) = J(b)$.

Similarly

$$N \supset \{c \in C;\ J^-(c) > -\infty\}.$$

Let $x \in L$. Then to any $\varepsilon > 0$ there is $b \in B$ such that $b \geqq x$ and

$$J(x) + \varepsilon > J(b) = I(b) = I(x)$$

hence

$$J(x) \geqq I(x).$$

Similarly there is $c \in C$ such that $c \leqq x$ and

$$J(x) - \varepsilon < J(c) = I(c) \leqq I(x),$$

hence

$$J(x) \leqq I(x).$$

Therefore $J(x) = I(x)$ for every $x \in L$.

2. Subadditive measures

A subadditive measure is a subadditive non-negative set function μ defined on a ring and upper semicontinuous in $\emptyset$, $\mu(\emptyset) = 0$. It can be easily proved that μ is upper and lower semicontinuous in any set and therefore it is also σ-subadditive.

Let S be the family of all subsets of a set X. S is partially ordered by inclusion. If we introduce $A + B = A \cup B$ as set-theoretical union and $A - B$ as set-theoretical difference, then S satisfies all assumptions of § 1. Let $A \subset S$ be such a ring that X can be covered by a sequence of sets of A. Let μ be a finite subadditive measure on A. The axioms (i), (ii), (iii) and (iv) are also evidently satisfied. The axiom (v) is equivalent to the following one:

If $E_n \in A$ $(n = 1, 2, \ldots)$, E_n are pairwise disjoint and $\lim_{n\to\infty} (\bigcup_{i=1}^{n} E_i) < \infty$, then $\lim_{n\to\infty} \mu(E_n) = 0$.

If the condition is satisfied, then according to the Theorem, μ can be extended to a subadditive measure on the monotone system generated by A and this system contains the δ-ring generated by A.

The result is similar to a result by L. Drewnowski ([4], Theorem 7.2). Drewnowski extends a subadditive measure (in his terminology an order continuous submeasure) from a ring to the generated σ-ring, but his additional condition is a little more strict ($E_n \in A, E_n$ disjoint $\Rightarrow \lim_{n\to\infty} \mu(E_n) = 0$).

From our Theorem there follows the main result of paper [5] by V. N. Alexiuk and F. D. Beznosikov (theorem 5.1). It suffices to put in our theorem S equal to a Boolean σ-algebra and A equal to a subalgebra of S ($a + b = a \cup b$, $a - b = a \cap b'$). The additional condition of Alexiuk and Beznosikov is the same as the Drewnowski condition, hence it is a little more strict than our condition (v). (Note that in [5] a subadditive measure is called a continuous outer measure.)

3. Subadditive integral

Let S be the a linear, conditionally σ-complete lattice of real-valued functions on a set X. Let $A \subset S$ be such a linear lattice that to any $f: X \to R$ there are $f_n, g_n \in A$ $(n = 1, 2, \ldots)$ with $\inf_n f_n \leqq f \leqq \sup_n g_n$. Let $J_0: A \to R$ be a function satisfying the conditions:

(i) $J_0(f + g) \leqq J_0(f) + J_0(g)$ for all $f, g \in A$.

(ii) $f, g \in A$, $f \leqq g \Rightarrow J_0(f) \leqq J_0(g)$.

(v) If $f_n \leqq f_{n+1} \leqq f \in S$, $f_n \in A$ $(n = 1, 2, \ldots)$ and $\{J_0(f_n)\}_{n=1}^{\infty}$ is bounded, then $\lim_{n\to\infty} J_0(f_{n+1} - f_n) = 0$.

Evidently we can use our general scheme and extend J_0 to a "full subadditive" integral. The property (iii) is satisfied automatically since,

$$J_0(b) = J_0((b - a) + a) \leqq J_0(b - a) + J_0(a),$$

white (iv) follows immediately from (v) and (ii).

Here we present only an example. Let $\mathscr{S}$ be a σ-algebra of subsets of a set X, μ be a finite subadditive measure on $\mathscr{S}$, A be the set of all functions $f: X \to R$ which can be expressed in the form $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, $E_i \in \mathscr{S}$, $\alpha_i \in R$, n positive integer. Put

$$J_0(f) = \inf \{\sum_{i=1}^{n} \alpha_i \mu(E_i);\ f \leqq \sum_{i=1}^{n} \alpha_i \chi_{E_i}\}.$$

Obviously (i) and (iii) are satisfied.

Let $f_n \in A$, $g_n \in S$, $0 \leqq f_n \leqq g_n, g_n \searrow O$. Put $h_n = \sup \{f_i; i \geqq n\}$. Then $O \leqq f_n \leqq h_n \leqq g_n$, $h_n \searrow O$. Put $G_n = \{x \in X;\ h_n(x) \geqq \varepsilon\}$, where ε is an arbitrary positive number. Then $G_n \in \mathscr{S}$, $G_n \supset G_{n+1}$ $(n = 1, 2, \ldots)$, $\bigcap\limits_{n \to 1}^{\infty} G_n = \emptyset$, hence $\lim\limits_{n\to\infty} \mu(G_n) = 0$. Finally, let $M = \sup h_1 < \infty$. We have

$$J_0(f_n) \leqq J_0(f_n \chi_{G_n}) + J_0(f_n \chi_{X-G_n}) \leqq$$
$$\leqq M\mu(G_n) + \varepsilon\mu(X - G_n) \leqq M\mu(G_n) + \varepsilon\mu(X) .$$

It follows that

$$0 \leqq \limsup_{n\to\infty} J_0(f_n) \leqq M \lim_{n\to\infty} \mu(G_n) + \varepsilon\mu(X) = \varepsilon\mu(X)$$

for every $\varepsilon > 0$, hence

$$\lim_{n\to\infty} J_0(f_n) = 0 .$$

Remark. Additivity can also be expressed in the general case:

$$J_0(a) + J_0(b) = J_0(a \cup b) + J_0(a \cap b);$$
$$a \leqq b \Rightarrow J_0(b) = J_0(a) + J_0(b - a) .$$

Now we should prove that also J is "additive". But corresponding results are known and they hold under weaker assumptions about $\mathscr{S}$ (see [3]).

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