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I. Introduction

In [1] J. Malkevitch formulated the following problem: for which n ($n \geqq 5$) does there exist a planar graph G_j with n vertices such that G_j contains cycles of every length m for $3 \leqq m\,(\neq j) \leqq n$ with $4 \leqq j \leqq n-1$. In [2] a slightly more general problem is solved. In Part III of this paper an analogous result will be proved without the assumption of planarity. Part IV is devoted to solving a similar problem for digraphs.

II. Notions and symbols

A path of length n in a graph G is a finite sequence

$$s = (v_0, x_1, v_1, x_2, \dots, v_{n-1}, x_n, v_n)\,,$$

where v_i ($i = 0, 1, \dots, n$) are vertices of G and x_j ($j = 1, 2, \dots, n$) is an edge connecting v_{j-1} to v_j. If G has no multiple edges, we shall write

$$s = (v_0, v_1, \dots, v_{n-1}, v_n)\,.$$

A path s is closed if $v_0 = v_n$ and it is a cycle if all the vertices are distinct (except $v_0 = v_n$) and $n \geqq 3$.

Analogously a directed path in a digraph G is a sequence

$$s = (v_0, x_1, v_1, x_2, v_2, \dots, x_n, v_n)\,,$$

where v_i are vertices of G and x_j is a directed edge from v_{j-1} to v_j. We shall call s a directed cycle of length n if $n \geqq 3$, $v_0 = v_n$ and the vertices $v_1, v_2, v_3, \dots, v_n$ are distinct.

Consider natural numbers n and j such that $3 \leqq j \leqq n$.

Definition. *A graph (digraph) G with n vertices is*

a) *pancyclic if it contains cycles (directed cycles) of every length m for $3 \leqq m \leqq n$,*

b) *j-pancyclic if it contains cycles (directed cycles) of every length m for* $3 \leqq m \leqq n$ *and* $m \neq j$.

III. Generalization of a theorem concerning j-pancyclic graphs

In [2], where a j-pancyclic graph was assumed to be planar in every case, the problem stated by J. Malkevitch in [1] is solved. The solution has taken the form of the following theorem:

Theorem 1. *If* $(n, j) \in \{(5, 3), (5, 4), (6, 3), (6, 5)\}$, *then there exists no j-pancyclic planar graph with n vertices. For any other combination of n and j the always exists a j-pancyclic planar graph with n vertices.*

We shall now show that the planarity assumption may be omitted. This will be done by proving the following theorem:

Theorem 2. *Let n and j be natural numbers such that* $3 \leqq j \leqq n$. *If* $(n, j) \in \{(5, 3), (5, 4), (6, 3), (6, 5)\}$, *then there exists no j-pancyclic graph with n vertices. For any other value of n and j there always exists a j-pancyclic graph with n vertices.*

Proof. To prove the non-existence of a j-pancyclic graph with n vertices for $(n, j) \in \{(5, 3), (5, 4), (6, 3), (6, 5)\}$ we have only to consider that if such a graph existed it would have to contain (in addition to Hamiltonian cycles) cycles of length 4, 3, 5 or 3, respectively, which easily leads to a contradiction. To complete the proof, it is only necessary to prove, for all other pairs (n, j), the existence of a j-pancyclic planar graph with n vertices, which has been done already in the proof of the corresponding theorem in [2].

The problem which was examined in [2] would have a more interesting solution if the graphs were restricted to being 3-connected.

IV. j-pancyclic digraphs

Theorem 3. *Let* n, j *be natural numbers such that* $3 \leqq j \leqq n$. *Then there exists a j-pancyclic digraph with n vertices.*

Proof. For $3 \leqq n \leqq 4$ the conclusion is trivial, for $n = 6$ and $j = 4$ it is proved by fig. 1. For $n = 5$, $3 \leqq j \leqq 5$; $n = 6$, $j = 3, 5, 6$ and $n \geqq 7$, $3 \leqq j \leqq n$ we shall prove the theorem by constructing a j-pancyclic graph with n vertices. In doing so, we shall have to distinguish two cases.

A. Let $3 \leqq j < n$.

Construct a directed cycle of length n and call its vertices, in the following order, $v_1, v_2, \ldots, v_{j-1}, v_j, \ldots, v_n$. Then construct the remaining directed edges as follows:

(a) If $\left[\frac{n}{2}\right] + 1 < j \leqq n - 1$, connect by a directed edge v_1 to v_j; then add directed edges (v_q, v_1) for $3 \leqq q \leqq j - 1$, and (v_j, v_r) for $j + 2 \leqq r \leqq n$. A directed graph constructed in this way (see fig. 2) contains no directed cycle of length j but does contain a cycle of length m for every m such that $3 \leqq m \leqq n$ and $m \neq j$.

Each such cycle is constructed

1. either from some of the vertices $v_1, v_2, \ldots, v_{j-1}$,
2. or from some of the vertices $v_j, v_{j+1}, \ldots, v_n, v_1$,
3. or by extending a directed path $s = (v_1, v_2, \ldots, v_j)$ along the directed edges (v_j, v_q) for $j + 2 \leqq q \leqq n$. In the first case the cycle has the form

$$(v_1, v_2, \ldots, v_i, v_1) \quad \text{with} \quad i < j$$

In the second case the corresponding form is

$$(v_k, \ldots, v_n, v_1, v_j, v_k) \quad \text{for} \quad k > j$$

and the cycle has length $\leqq n - j + 2$ which, since $\left[\frac{n}{2}\right] + 1 < j$, does not

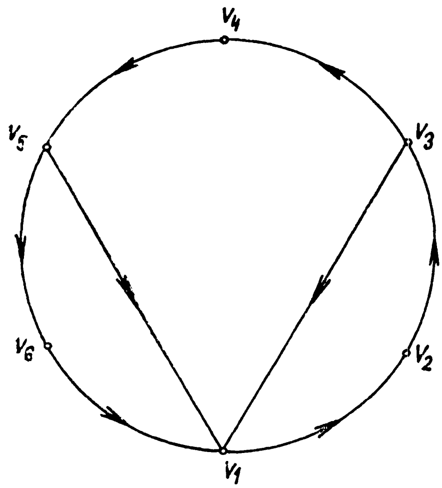

Fig. 1

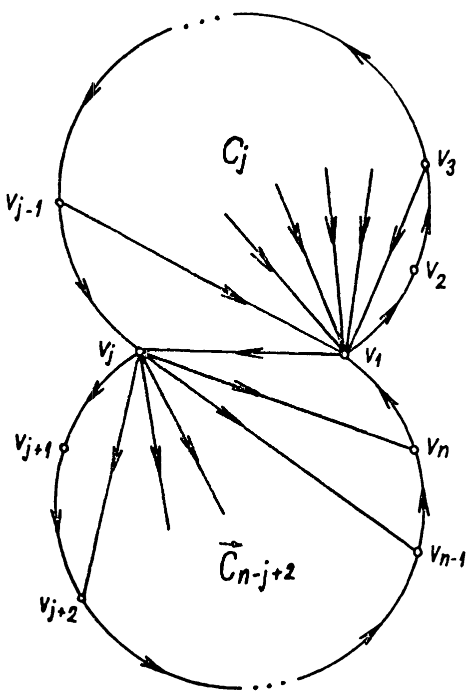

Fig. 2

exceed $j - 1$. In the third case the length of the cycle is greater than j, since extending a directed path s to pass over any one of the vertices v_j, $v_{j+1}, \ldots, v_n$ can only lead to a cycle whose length is at least $j + 1$. Thus the graph contains no cycle of length j.

It remains to prove that it does contain cycles of length m for $3 \leqq m \leqq$ $\leqq n$ and $m \neq j$. For $3 \leqq m \leqq j - 1$ the cycle

$$(v_1, v_2, v_3, \ldots, v_m, v_1)$$

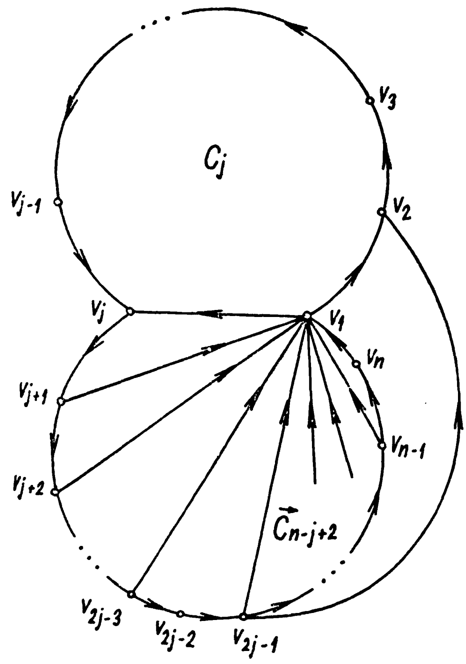

Fig. 3

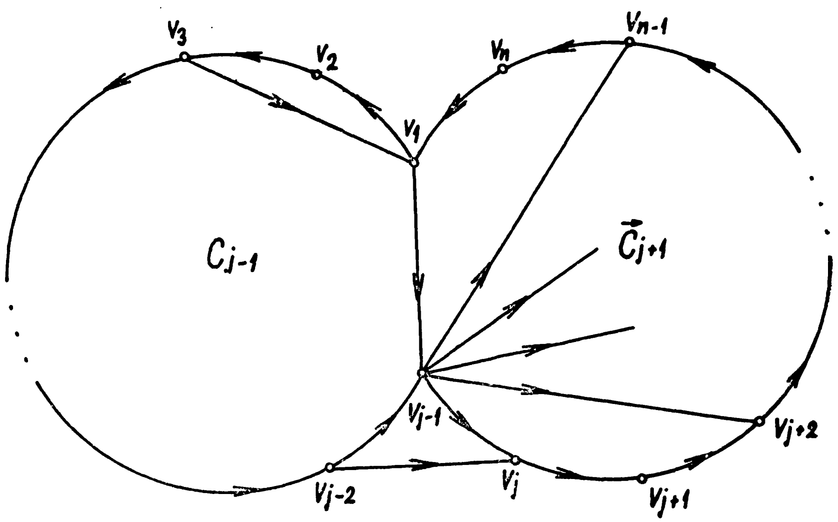

Fig. 4

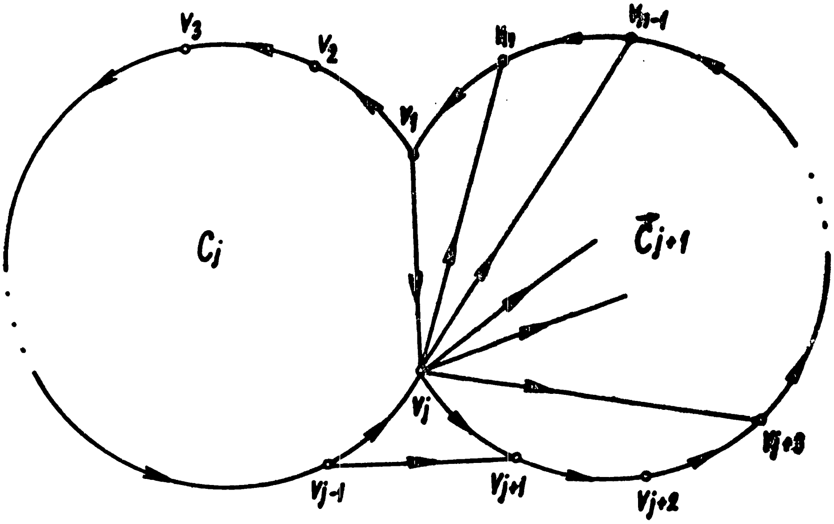

Fig. 5

is the required one; for $j + 1 \leqq m \leqq n$ the required cycle is

$$(v_1, v_2, \ldots, v_j, v_{j+n+1-m}, \ldots, v_n, v_1)\,.$$

(b) If $j < \left[\dfrac{n}{2}\right] + 1$, we construct a directed edge connecting the vertex v_1 to v_j. We proceed to add directed edges (v_q, v_1) for $j + 1 \leqq q \leqq n - 1$, $q \neq 2j - 2$. In addition to this, we connect v_{2j-1} to v_2. Such a directed graph (see fig. 3) contains no cycles of length j and contains a cycle of length m for every m such that $3 \leqq m \leqq n$ and $m \neq j$. The non-existence of a cycle of length j and the existence of the other cycles is shown by a reasoning similar to that of (a).

(c) For $j = \left[\dfrac{n}{2}\right] + 1$ we shall distinguish two cases:

1. If $n > 6$, $n \equiv 0 \pmod 2$, then we connect v_1 to v_{j-1} by a directed edge, likewise v_3 to v_1 and v_{j-1} to v_q for $j + 2 \leqq q \leqq n - 1$. In addition to this, we connect v_{j-2} to v_j (see fig. 4). In such a directed graph it is easy to verify that there is no cycle of length j while there is a cycle of length m for $3 \leqq$ $\leqq m\ (\neq j) \leqq n$.

2. For $n \geqq 5$, $n \equiv 1 \pmod 2$ we connect v_1 to v_j and v_j to v_q for $j + 3 \leqq$ $\leqq q \leqq n$. In addition to this we connect v_{j-1} to v_{j+1}. A digraph constructed in this way (see fig. 5) is easily seen to contain no cycle of length j while at the same time containing a cycle of length m for all m such that $3 \leqq m \leqq n$ and $m \neq j$.

B. Let $j = n$.

Construct a cycle of length n-1 and call its vertices, in sequence, v_1, $v_2 \ldots$ v_{n-1}. Add another vertex v_n which does not belong to the cycle. Add directed adges (v_q, v_1) for $3 \leqq q \leqq n$, $q \neq n - 1$ and also the directed edge (v_n, v_{n-1}).

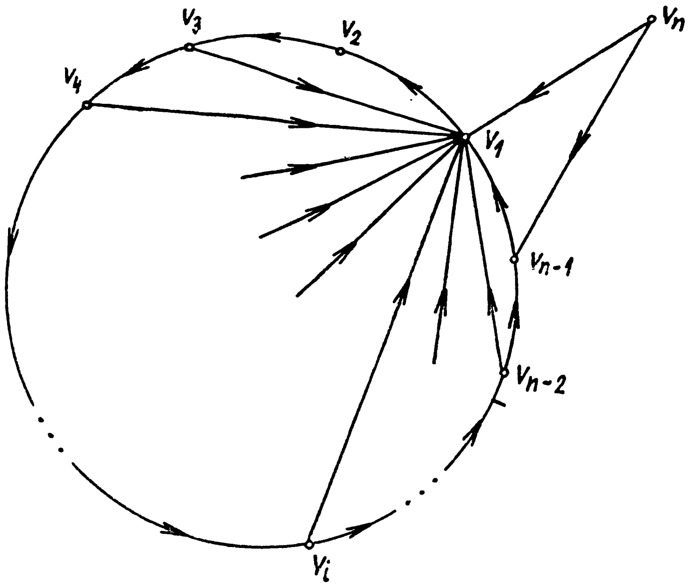

Fig. 6

Evidently this graph (see fig. 6) contains no cycle of length j but does contain a cycle of length m for all m such that $3 \leqq m \leqq n - 1$.

This completes the proof of our theorem.

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