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PROJECTIVE TENSOR PRODUCT OF l^p-VALUED MEASURES

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In [2], M. Duchoň considers the projective tensor product of two vector-valued measures. That is, if $\mu: \mathscr{M} \to X$ and $\nu: \mathscr{N} \to Y$ are countably additive set functions with values in the Hausdorff locally convex topological vector spaces X and Y, then the product measure $\mu \times \nu$ is defined on a measurable rectangle $A \times B$ by $\mu \times \nu\ (A \times B) = \mu(A) \otimes \nu(B)$, where $x \otimes y$ denotes the tensor product of x and y. If $\mathscr{A}$ is the algebra of sets generated by the measurable rectangles $A \times B$ where $A \in \mathscr{M}$ and $B \in \mathscr{N}$, then $\mu \times \nu$ has a unique finitely additive extension to a mapping from $\mathscr{A}$ into the tensor product $X \otimes Y$. If $\sum$ is the σ-algebra generated by $\mathscr{A}$, then the locally convex space X is an **admissible factor** if for any locally convex space Y and any pair of vector measures $\mu: \mathscr{M} \to X$ and $\nu: \mathscr{N} \to Y$, $\mu \times \nu$ is countably additive on $\mathscr{A}$ with respect to the projective topology (π-topology) on $X \hat{\otimes}_\pi Y$ ([7] § 43) and has a (necessarily unique) countably additive extension from $\sum$ into $X \hat{\otimes}_\pi Y$ ([2]). Duchoň notes ([2], [4]) that any nuclear space is an admissible factor.

In this note we present measures $\mu: P(N) \to l^p\ (1 \leq p < \infty)$ and $\nu: P(N) \to c_0$ such that the product $\mu \times \nu$ is not countably additive on $\mathscr{A}$. (Here $P(N)$ denotes the power set of the natural numbers N.) Thus no l^p space $(1 \leq p < \infty)$ is an admissible factor; in particular, l^1 is not an admissible factor. The example showing that l^1 is not an admissible factor reveals an error in the proof of an assertion in [2] (see also [3]) that l^1 is an admissible factor. For another example constructed using the Dvoretsky—Rogers Theorem see [8]; see also remark 5 below. We also give necessary and sufficient conditions in order that a fixed l^1-valued measure μ and any c_0 valued vector measure ν be such that $\mu \times \nu$ has countably additive extension from $\sum$ into $l^1 \hat{\otimes}_\pi c_0$.

A vector measure $\mu: \mathscr{M} \to X$($\mathscr{M}$ a σ-algebra) admits products with respect to Y if for every vector measure $\nu: \mathscr{N} \to Y$($\mathscr{N}$ a σ-algebra) the product $\mu \times \nu$ has a countably additive extension from $\sum$, the σ-algebra generated by measurable rectangles, into $X \hat{\otimes}_\pi Y$. Thus X is an admissible factor if every X-valued vector measure admits products with respect to every locally convex space Y.

In Theorem 5 we present a necessary and sufficient condition for a fixed l^1-valued measure to admit products with respect to c_0.

In Lemma 1 and Theorem 2 below, $\mathscr{A}$ will denote the algebra of subsets of $N \times N$ generated by the rectangles $A \times B$, $A \subseteq N$, $B \subseteq N$.

Lemma 1. *Let $\{x_n\}_{n=1}^{\infty}$ (with $x_n = \{x_{nm}\}_{m=1}^{\infty}$) be unconditionally convergent in l^p $(1 \leq p < \infty)$ and $\{s_n\}_{n=1}^{\infty} \in c_0$. Suppose the vector measure $\mu : P(N) \to l^p$ is defined by $\mu(A) = \sum_{n \in A} x_n$ and ν: $P(N) \to c_0$ is defined by $\nu(A) = \sum_{n \in A} s_n e_n$, where e_n is the sequence whose only non-zero entry is a 1 in the n-th position. If $\mu \times \nu$ is countably additive from $\mathscr{A}$ into $l^p \hat{\otimes}_{\pi} c_0$, then for each $\{\alpha_n\}_{n-1}^{\infty}$ in $l^{p'}$ $(1/p + 1/p' = 1)$ we have $\sum_{m,n} |\alpha_n x_{mn} s_n| < \infty$.*

Proof. If $\mu \times \nu$ is countably additive on $\mathscr{A}$, then $\sum_{m,n} \mu \times \nu(m, n)$ is unconditionally convergent in $l^p \hat{\otimes}_{\pi} c_0$, and thus $\sum_{m,n} | < \alpha, \mu \times \nu(m, n) > | < \infty$ for each α in the dual of $l^p \hat{\otimes}_{\pi} c_0$.

Regarding $e_n \in l^1 = c_0^*$, we may interpret $\alpha = \sum_{n=1}^{\infty} \alpha_n e_n$ as an element of $(l^p \hat{\otimes}_{\pi} c_0)^*$ ([7], 43.4). Then $< \alpha, \mu \times \nu(m, n) > = \sum_{l=1}^{\infty} \alpha_l x_{ml} \delta_{ln} s_n = \alpha_n x_{mn} s_n$, and the lemma is proved.

For our example, we first construct an example of a series in l^p $(1 \leq p < \infty)$ which is unconditionally convergent but not absolutely convergent.

Define a sequence of matrices $\{\mathbf{A}_j\}_{j=0}^{\infty}$ by $\mathbf{A}_0 = [1]$ and $\mathbf{A}_{j+1} = \begin{bmatrix} \mathbf{A}_j & \mathbf{A}_j \\ -\mathbf{A}_j & \mathbf{A}_j \end{bmatrix}$. Obviously $\mathbf{A}_j$ is a 2^j by 2^j matrix; a simple inductive argument shows that distinct rows of $\mathbf{A}_j$ are orthogonal for each fixed j. For the inductive step, note that the inner product of two rows of $\mathbf{A}_j$ has the form $\pm a \,.\, b + a \,.\, b$, where a and b are rows of $\mathbf{A}_{j-1}$.

Choose t satisfying

$$2^{-1-1/p} < t < \min (2^{-1}, 2^{-\frac{1}{2}-1/p}) \tag{1}$$

and let $\mathbf{A}$ be the infinite matrix

$$\begin{bmatrix} \mathbf{A}_0 & 0 & 0 & 0 \dots \\ 0 & t\mathbf{A}_1 & 0 & 0 \dots \\ 0 & 0 & t^2\mathbf{A}_2 & 0 \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Let $x_n = \{x_{nk}\}_{k=1}^{\infty}$ be the n-th row of $\mathbf{A}$. We show that $\sum_{n=1}^{\infty} x_n$ has the desired properties.

Let $I_j = \{2^j, 2^j + 1, \dots, 2^{j+1} - 1\}$. For each n there is a unique j such that

$n \in I_j$; we see that $x_{nk} = \pm t^j$ for $k \in I_j$ and $x_{nk} = 0$ for $k \notin I_j$. Thus $||x_n||_p^p = 2^j t^{pj}$ for $n \in I_j$ so that

$$\sum_{n=1}^{\infty} ||x_n||_p = \sum_{j=0}^{\infty} \sum_{n \in I_j} ||x_n||_p = \sum_{j=0}^{\infty} 2^j \, 2^{j/p} t^j = \infty \text{ by (1).}$$

Since the rows of each $\mathbf{A}_j$ are orthogonal we see

$$\sum_{k=1}^{\infty} x_{nk} x_{mk} = \begin{cases} 0, n \neq m \\ 2^j t^{2j}, n = m \in I_j \end{cases} \tag{2}$$

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence in $\{0, 1\}$ with at most finitely many non-zero terms. Define $y = \{y_k\}_{k=1}^{\infty}$ as the finite sum $y = \sum_{n=1}^{\infty} \varepsilon_n x_n$. For $k \in I_j$ the sum for y_k reduces to $\sum_{n \in I_j} \varepsilon_n x_{nk}$.

Note that y represents an arbitrary finite partial sum of an arbitrary rearrangement of $\sum_{n=1}^{\infty} x_n$ or the difference of two such partial sums.

First we consider the case $1 \leq p \leq 2$. Let J be the first integer j for which there is an $n \in I_j$ with $\varepsilon_n \neq 0$. We have

$$||y||_p^p = \sum_{k=1}^{\infty} |y_k|^p = \sum_{j=J}^{\infty} \sum_{k \in I_j} |y_k|^p \leq \sum_{j=J}^{\infty} 2^{j(1-p/2)} \, (\sum_{k \in I_j} |y_k|^2)^{p/2} \tag{3}$$

by Hölder's inequality. Since

$$\sum_{k \in I_j} |y_k|^2 = \sum_{k \in I_j} (\sum_{n \in I_j} \varepsilon_n x_{nk})^2 = \sum_{k \in I_j} \sum_{m \in I_j} \sum_{n \in I_j} \varepsilon_m \varepsilon_n x_{mk} x_{nk} = \sum_{n \in I_j} \varepsilon_n^2 \, 2^j \, t^{2j} \leq 2^j \, . \, 2^j \, t^{2j}$$

by (2),
we have from (3)

$$||y||_p^p \leq \sum_{j=J}^{\infty} 2^{j(1-p/2)} \, . \, 2^{pj} \, t^{pj} = \sum_{j=J}^{\infty} 2^{j(1+p/2)} \, t^{pj}. \tag{4}$$

By (1), $\sum_{j=0}^{\infty} 2^{j(1+p/2)} \, t^{pj} < \infty$; thus (4)

implies $\sum_{n=1}^{\infty} x_n$ is unconditionally Cauchy in l^p

and $\quad ||\sum_{n-1}^{\infty} x_n||_p^p \leq \sum_{j=0}^{\infty} 2^{j(1+p/2)} \, t^{pj} \quad$ for $\quad 1 \leq p \leq 2$.

When $2 < p < \infty$, the above arguments show that $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent in l^2 and thus $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent in l^p as well.

Theorem 2. *For $1 \leq p < \infty$, define a vector measure $\mu: P(N) \to l^p$ by $\mu(A) = \sum_{n \in A} x_n$, where $x_n = \{x_{nk}\}_{k=1}^{\infty}$ is as constructed. Define* $\{s_n\}_{n=1}^{\infty} \in c_0$ by $s_n = (j+1)^{-1}$ *for $n \in I_j$, and define a vector measure ν: $P(N) \to c_0$ by $\nu(A) = \sum_{n \in A} s_n e_n$. Then $\mu \times \nu$ is not countably additive from $\mathscr{A}$ into $l^p \hat{\otimes}_\pi c_0$.*

Proof. Suppose $\mu \times \nu$ is countably additive. Then by lemma 1, $\sum_{m,n} |\alpha_n x_{mn} s_n| < \infty$ for every $\{\alpha_n\}_{n=1}^{\infty}$ in $l^{p'}$. Define $\alpha_n = 2^{-2j} t^{-j}$ for each $n \in I_j, j = 0, 1, 2, \ldots$. For $p = 1$, clearly $\alpha = \{\alpha_n\}_{n=1}^{\infty} \in l^\infty = (l^1)^*$; in the case $1 < p < \infty$ we compute

$$||\alpha||_{p'}^{p'} = \sum_{j=0}^{\infty} \sum_{n \in I_j} |\alpha_n|^{p/(p-1)} = \sum_{j=0}^{\infty} \sum_{n \in I_j} (2^{-2j} t^{-j})^{p/(p-1)} = \sum_{j=0}^{\infty} 2^j (2^{-2j} t^{-j})^{p/(p-1)} .$$

Since $2(2^{-2} t^{-1})^{p/(p-1)} < 1$ by (1), this series is convergent.

But we have

$$\sum_{m,n} |\alpha_n x_{mn} s_n| = \sum_{j=0}^{\infty} \sum_{m \in I_j} \sum_{n \in I_j} |\alpha_n x_{mn} s_n| = \sum_{j=0}^{\infty} \sum_{m \in I_j} \sum_{n \in I_j} 2^{-2j} t^{-j} \,.\, t^j \,.\, (j+1)^{-1}$$
$$= \sum_{j=0}^{\infty} (j+1)^{-1} = \infty, \quad \text{a contradiction.}$$

Remark 3. From the examples above and the results of [2], it is natural at this point to conjecture that a locally convex space X is an admissible factor iff X is nuclear.

Remark 4. If we set $y_n = s_n e_n$ as in Lemma 1, then we have shown that the series $\sum_{n,m} x_n \hat{\otimes} y_m$ is not weakly unconditionally Cauchy in $l^p \hat{\otimes}_\pi c_0$ for $1 \leq p < \infty$ (see the proof of Lemma 1). In particular this implies that $\mu \times \nu$ is not even bounded on $\mathscr{A}$. In [5] Kluvanek has also constructed an example of such a pair of series and noted its significance with respect to projective tensor products of vector measures. The examples presented above are much simpler than Kluvanek's example although his example does take place in reflexive Banach spaces.

M. Duchoň ([3]) shows that if μ is a vector measure of bounded variation, then μ admits products with respect to any locally convex space. Using Lemma 1 we establish the converse of this result for l^1-valued vector measures. In fact, we have

Theorem 5. *If $\mu: \mathscr{M} \to l^1$, $\mathscr{M}$ a σ-algebra, admits products with respect to c_0, then μ has bounded variation.*

Proof. Let $\{A_m\}$ be a disjoint sequence from $\mathscr{M}$ and set $\mu(A_n) = x_n = \{x_{ni}\}_{i=1}^{\infty}$. Since $\sum x_n$ is unconditionally convergent in l^1, it follows from Lemma 1 that $\sum_{n=1}^{\infty} |s_n| \sum_{m=1}^{\infty} |x_{mn}| < \infty$ (take $\alpha_n = 1$ for each n) for every $\{s_n\} \in c_0$. Since $\{s_n\} \in c_0$ is arbitrary, $\sum_{n,m=1}^{\infty} |x_{nm}| = \sum_{n=1}^{\infty} ||x_n|| = \sum_{n=1}^{\infty} ||\mu(A_n)|| < \infty$. Hence, μ has bounded variation ([1] Th. 4).

Remark 5. It, of course, follows directly from Theorem 5 and the Dvoretsky—Rogers Theorem ([7]) that l^1 is not an admissible factor. However, the examples presented above actually exhibit two vector measures whose projective tensor product does not have a countably additive extension and, thus, also shows that c_0 and l^p are not admissible factors.

Theorem 5 also shows that the Corollary of [3] is the best general result, that can be expected for measures that admit products, i. e., a vector measure $\mu : \mathscr{M} \to l^1$ admits products with respect to c_0 iff μ has bounded variation (and iff μ is dominated with respect to c_0 ([3]).

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