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A COMPACTNESS PROPERTY OF FOURIER-STIELTJES TRANSFORMS

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Let G be a locally compact Abelian group and Γ its dual group. Denote by $B(G)$ the set of Fourier-Stieltjes transforms on G, i. e. of functions f of the form

$$f(x) = \int_{\Gamma} (x, \gamma)\, \mu(\mathrm{d}\gamma), \quad x \in G\,, \tag{1}$$

where μ is a regular measure on $\mathscr{B}(\Gamma)$, the system of Borel sets in Γ, and (x, γ) denotes the value of $\gamma \in \Gamma$ on $x \in G$. For a function f on G and $y \in G$, f_y denotes the function defined by $f_y(x) = f(x + y)$, $x \in G$. Finally, $C(G)$ will stand for the set of bounded continuous functions on G considered also as a Banach space with respect to sup norm.

Theorem. *A function f on G belongs to $B(G)$ if and only if the set of functions*

$$\sum_{i=1}^{n} c_i f_{y_i}\,, \tag{2}$$

formed for all choices of $y_i \in G$ and complex numbers c_i and for $n = 1, 2, \ldots$ such that

$$\sup_{\gamma \in \Gamma} \left| \sum_{i=1}^{n} c_i (y_i, \gamma) \right| \leqslant 1\,, \tag{3}$$

is relatively weakly compact subset of $C(G)$.

For the weak compactness of the set in question, it is obviously necessary that, for every $x \in G$, the set of numbers $\Sigma c_i f_{y_i}(x)$ be bounded. But there is a result by W. F. Eberlein [1] stating that the boundedness of the set of numbers $\Sigma\, c_i f(y_i)$, subject to the condition (3), is necessary and sufficient for $f \in C(B)$ to be in $B(G)$. Hence the Theorem can be expressed in the following form:

Given $f \in C(G)$, the set of functions (2), such that the condition (3) is satisfied, is relatively weakly compact in $C(G)$ if and only if there is a point in G in which the set of their values is bounded.

The proof of the Theorem can be based on the following proposition proved in [2; Theorem 3]:

Given a Banach space X, a function $\Phi : G \to X$ can be represented in the form

$$\Phi(y) = \int_\Gamma (y, \gamma)\, m(\mathrm{d}\gamma), \quad y \in G ,$$

where m is a regular X-valued measure on $\mathscr{B}(\Gamma)$, if and only if the set of vectors $\Sigma c_i \Phi(y_i)$, for $y_i \in G$, complex c_i and $n = 1, 2, \ldots,$ satisfying (3), *is relatively weakly compact in X.*

In fact, choosing $X = C(G)$ and $\Phi(y) = f_y$, we see that the Theorem is a direct consequence of the following.

Lemma. *Given a complex-valued function f on G, there exists a regular $C(G)$-valued measure m on $\mathscr{B}(\Gamma)$ such that*

$$f_y = \int_\Gamma (y, \gamma)\, m(\mathrm{d}\gamma), \quad y \in G , \tag{4}$$

if and only if $f \in B(G)$.

Proof. Suppose $f \in B(G)$ and put

$$m(E)\,(x) = \int_E (x, \gamma)\, \mu(\mathrm{d}\gamma), \quad x \in G ,$$

for every $E \in \mathscr{B}(\Gamma)$, μ being as in (1). Since

$$|m(E)\,(x)| \leqslant \int_E |\mu|\,(\mathrm{d}\gamma) = |\mu|\,(E) ,$$

it is immediate that m is a regular σ-additive $C(G)$-valued function on $\mathscr{B}(\Gamma)$. It follows further that, for every $y \in G$, the function $\gamma \to (y, \gamma)$ is m-integrable and, since

$$f_y(x) = f(x + y) = \int_\Gamma (x + y, \gamma)\, \mu(\mathrm{d}\gamma) = \int_\Gamma (y, \gamma)\,(x, \gamma)\, \mu(\mathrm{d}\gamma) =$$

$$= \int_\Gamma (y, \gamma)\, m(\mathrm{d}\gamma)\,(x)$$

uniformly with respect to $x \in G$, we have (4).

If, on the other hand, (4) holds, we put $\mu(E) = m(E)\,(0)$, for every $E \in \mathscr{B}(\Gamma)$. Then

$$f(y) = f_y(0) = \int_\Gamma (y, \gamma)\, m(\mathrm{d}\gamma)\,(0) = \int_\Gamma (y, \gamma)\, \mu(\mathrm{d}\gamma), \quad y \in G ,$$

which is (1).

Note that, if m is such that (4) holds, then m has finite variation, i. e. there is a finite non-negative measure ν such that $\|m(E)\| \leqslant \nu(E)$, for $E \in \mathscr{B}(\Gamma)$. Further, m has a density with respect to ν, i. e. $m(E) = \int_E \Phi\, \mathrm{d}\nu$, $E \in \mathscr{B}(\Gamma)$, where Φ is a $C(G)$-valued function on Γ. Neither of these properties hold in general for a $C(G)$-valued measure on $\mathscr{B}(\Gamma)$.

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