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NOTE ON STOCHASTIC TRANSFORMERS

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On many practical occassions we meet a source (emitter) of random signals which are processed by some random or non-random devices. Of course the words „source“, „signals“ (= letters from a finite alphabet), „device“ have a very general meaning; e. g. the „source“ may be a railway which emits tank trains to a factory, etc.

Both the source and the processor are usually well described but their general behaviour together is not always easy to predict or calculate. Sometimes the Monte Carlo method may be (or may seem to be) the most suitable and useful method of solution for the above problem. The main purpose of this paper is to show the possibility of a stationary source simulation by a special type of stochastic automaton. Similarly as in Gill's fundamental paper [1] and in [2], [3] we shall also consider the approximate simulation in the sense of some metrics, defined on the space of all stationary sources with a given alphabet. It is obvious that the metrics will depend on the „processing device“. However, metrics can be found which may be useful in many practical cases. In [1], [2], [3] the metric expresses the maximum difference between the probabilities of individual letters in both sources. This metric is suitable for independent sources only; for our purposes we shall define a metric which takes in account probabilities of all strings (the „global“ metric in the sense of [3]).

1. NOTATIONS AND DEFINITIONS

The symbols X, Y, Z denote finite alphabets, N denotes the set of all natural numbers.

The symbol X^N denotes the set of all sequences $x = (x_1, x_2, \ldots)$, where $x_i \in X$, $(i = 1, 2, \ldots)$. x is said to be periodical if there exists an $n \in N$ such that $x_{i+n} = x_i$ for $i = 1, 2, \ldots$

The set of all $x = (x_1, \ldots, x_n, \ldots) \in X^N$, such that for the given $i \in N$, $n \in N$, $a_1 \in X, \ldots, a_n \in X$ we have $x_{i+j} = a_j$ $(j = 1, \ldots, n)$ is called an elementary cylinder and denoted by $E^{i+1,\ldots,i+n}_{a_1,\ldots,a_n}$.

Instead of $E^{1,\ldots,n}_{a_1,\ldots,a_n}$ we use $E_{a_1,\ldots,a_n}$.

We denote by $\mathbf{X}^N$ the minimal α-algebra over the set of all elementary cylinders, X^* denotes the set of all $(x_1, \ldots, x_n) \in X \times \ldots \times X = X^n$ $(n \in N)$ and of the element $e \notin X$. The elements of X^* are called strings, e is the empty string.

The probability space $(X^N, \mathbf{X}^N, u)$ is called a source. We write $\mu(a_1, \ldots, a_n)$ instead of $\mu(E_{a_1,\ldots,a_n})$.

The source $\mathfrak{S} = (X^N, \mathbf{X}^N, \mu)$ is called stationary, if for every $(a_1, \ldots, a_n) \in X^n$ and $i \in N$ we have

$$\mu(a_1, \ldots, a_n) = \mu(E^{i+1,\ldots,i+n}_{a_1,\ldots,a_n}).$$

$\mathfrak{S}$ is called regular if for every $x = (x_1, x_2, \ldots) \in X^N$ we have $\mu(x) = \lim_{n\to\infty} \mu(x_1, \ldots, x_n) = 0$. The source $\mathfrak{S}$ is called degenerated if there exists an $x \in X^N$ such that $\mu(x) = 1$. If, moreover, x is periodical, $\mathfrak{S}$ is called periodically degenerated.

$\mathscr{A} = (A, X, Y, f, g, a_0)$ is called a generalised sequential machine (g.s.m.), if (see [4]) A, X, Y are finite sets of states, input and output signals, f is a mapping of $A \times X$ into A (a transmission function), g is the mapping of $A \times X$ into Y^* (an output function) and $a_0 \in A$ (the initial state). The mappings f and g can be extended to $A \times X^*$ in the following way:

$$f(a, x_1, \ldots, x_n) = f(\ldots(f(a, x_1), x_2, \ldots, x_n),$$

$$g(a, x_1, \ldots, x_n) = g(a, x_1), g(f(a, x_1), x_2), \ldots, g(f(a, x_1, \ldots, x_{n-1}), x_n).$$

If we replace Y^* by Y in the definition of g.s.m. we obtain the usual finite automaton.

The concept of the stochastic automaton without input (i.e. the stochastic generator) can be defined by generalization of the notion of a finite automaton but this way is not the most suitable. We prefer the method described e.g. in [5]: Let X, Z be finite alphabets, let $k \in N$ and let γ be a mapping of Z into X. Let $\mathfrak{T} = (Z^N, \mathbf{Z}^N, \pi)$ be a Markov source (Markov chain) with a transition matrix $(\pi(z_i/z_i)$ and with a given initial letter (initial state of the chain) z', i.e. let

$$\pi(z_1, \ldots, z_n) = \begin{cases} 0 \text{ for } z_1 \neq z' \\ \pi(z_2/z_1) \ldots \pi(z_n/z_{n-1}) \text{ otherwise.} \end{cases}$$

The source $\mathfrak{S} = (X^N, \mathbf{X}^N, \mu)$ is called an a-generator (a stochastic automaton without input) arising by composition of $\mathfrak{T}$ and γ at the origin k if we have

$$(x_1, \ldots, x_n) = \sum \pi(z_k, \ldots, z_{n+k-1})$$

where the sum is taken through all $(z_k, \ldots, z_{n+k-1})$ for which $(x_1, \ldots, x_n) = (\gamma(z_k), \ldots, \gamma(z_{n+k-1}))$.

If $\mathfrak{S} = (Y^N, \mathbf{Y}^N, \mu)$ and $\mathfrak{T} = (Y^N, \mathbf{Y}^N, \nu)$ are stationary sources, we put

$$\varrho(\mathfrak{S},\mathfrak{T}) = \sup |\mu(y_1, \ldots, y_n) - \nu(y_1, \ldots, y_n)|$$

where the supremum is taken through all strings $(y_1, \ldots, y_n) \in Y^*$. It is almost obvious that if $\mathbf{S}(Y)$ is the space of all stationary sources in the alphabet Y, then ϱ is a metric on $\mathbf{S}(Y)$.

Now we shall define a subspace of $\mathbf{S}(Y)$ from which we shall take the simulating (approximating) sources. For a fixed $\mathfrak{S} = (X^N, \mathbf{X}^N, \mu)$ and any $\mathscr{A} = (A, X, Y, f, g, a_0)$ we denote by $\mathfrak{T}(\mathfrak{S}, \mathscr{A}) = (Y^N, \mathbf{Y}^N, \nu)$ a source with the following property: We put $\nu(y_1, \ldots, y_n) = \sum \mu(x_1, \ldots, x_k)$ where the sum runs through all strings $(x_1, \ldots, x_k) \in X^*$ such that there exist $i \geqq 0$, $j \geqq 0$ for which we have $g(a_0, x_1, \ldots, x_k) = (y_1, \ldots, y_n, \ldots, y_{n+i})$ and $g(a_0, x_1, \ldots, x_{n-1}) = (y_1, \ldots, y_{n-1-j})$. The subspace $\mathbf{S}_{\mathfrak{S}}(Y)$ is defined as the set of all $\mathfrak{T}(\mathfrak{S}, \mathscr{A})$, where $\mathscr{A}$ is any g.s.m. with X, Y given.

2. REGULAR SOURCES

Lemma 1. *Let $\mathfrak{T} = (Y^N, \mathbf{Y}^N, \nu)$ be regular and $\varepsilon > 0$. Then there exists an integer $n(\varepsilon) \in N$ such that for every $(y_1, y_2, \ldots) \in Y^N$ we have $\nu(y_1, \ldots, y_{n(\varepsilon)}) < \varepsilon$.*

Proof. Indirectly. Suppose that for every $n \in N$ there exists a string $(y_1, \ldots, y_n)$ such that $\nu(y_1, \ldots, y_n) \geqq \varepsilon$. Denote

$$E_n = \bigcup_{(y_1,\ldots,y_n)\,:\,\nu(y_1,\ldots,y_n) \geqq \varepsilon} E_{y_1, \ldots, y_n}.$$

Obviously $E_{n+1} \subset E_n (n \in N)$ and $\nu(E_n) \geqq \varepsilon$, which implies that there exists an $y = (y_1, y_2, \ldots) \in \bigcap_{n \in N} E_n$ such that $\nu(y) = \lim_{n \to \infty} \nu(y_1, \ldots, y_n) \geqq \varepsilon$, which contradicts the regularity of the source.

Lemma 2. *Let $\mathfrak{T} = (Y^N, \mathbf{Y}^N, \nu)$ be a regular stationary source and $\varepsilon > 0$. Then there exists a stationary a-generator $\bar{\mathfrak{T}} = (Y^N, \mathbf{Y}^N, \bar{\nu})$ such that $\varrho(\mathfrak{T}, \bar{\mathfrak{T}}) > \varepsilon$.*

Proof. Let $n(\varepsilon)$ be defined for $\mathfrak{T}$ like in Lemma 1. Let $Z = \{e\} \cup Y \cup \ldots \cup Y^{n(\varepsilon)}$ and define the transition probabilities by the following requirements:

a) for $m < n(\varepsilon)$

$$\pi((y_1, \ldots, y_m, y_{m+1})/(y_1, \ldots, y_m)) = \frac{\nu(y_1, \ldots, y_{m+1})}{\nu(y_1, \ldots, y_m)},$$

b) for $m = n(\varepsilon)$

$$\pi((y_2, \ldots, y_m, y_{m+1})/(y_1, \ldots, y_m)) = \frac{\nu(y_1, \ldots, y_{m+1})}{\nu(y_1, \ldots, y_m)}.$$

Further in both cases a) and b) the transitions which are different from those just described have probabilities equal to zero.

Let us suppose that e (the empty string) is an initial state. Thus we have obtained a Markov source $(Z^N, \mathbf{Z}^N, \pi)$ with the essential states of the type $(y_1, \ldots, y_{n(\varepsilon)})$.

Now let us define the function γ for $z = e$ arbitrarily and for $z = (y_1, \ldots, y_m) \in Z, z \neq e$, by $\gamma(y_1, \ldots, y_m) = y_m$. Let $\bar{\mathfrak{X}} = (Y^N, \mathbf{Y}^N, \bar{\nu})$ be the α-generator which is a composition of $(Z^N, \mathbf{Z}^N, \pi)$ and γ at the origin 2. According to the definition of the α-generator

a) for $m \leqq n(\varepsilon)$ we have $\bar{\nu}(y_1, \ldots, y_m) = \nu(y_1, \ldots, y_m)$,

b) for $k \in N$ and $m = n(\varepsilon) + k$ we have

$$\bar{\nu}(y_1, \ldots, y_m) = \nu(y_1, \ldots, y_{n(\varepsilon)}) \cdot \frac{\nu(y_1, \ldots, y_{n(\varepsilon)+1})}{\nu(y_1, \ldots, y_{n(\varepsilon)})} \times \tag{1}$$

$$\cdot \frac{\nu(y_2, \ldots, y_{n(\varepsilon)+2})}{\nu(y_2, \ldots, y_{n(\varepsilon)+1})} \cdots \frac{\nu(y_k, \ldots, y_{n(\varepsilon)+k})}{\nu(y_k, \ldots, y_{n(\varepsilon)+k-1})}.$$

We shall prove that $\bar{\mathfrak{X}}$ is a stationary source. Let i, $m \in N$ and $(y_{i+1}, \ldots, y_{i+m}) \in Y^*$. Then

$$\nu(E^{i+1, \ldots, i+m}_{y+1, \ldots, y+m}) = \sum_{y_1 \in Y, \ldots, y_i \in Y} \bar{\nu}(y_1, \ldots, y_i, y_{i+1}, \ldots, y_{i+m})$$

and we have to show the last expression to be equal to

$$\bar{\nu}(y_{i+1}, \ldots, y_{i+m}) = \bar{\nu}(E^{1, \ldots, m}_{y_{i+1}, \ldots, y_{i+m}}).$$

For $i = 1$ this is trivial. If, for some i this statement is valid for all m and all $y_{i+1}, \ldots, y_{i+m}$, then for $i + m + 1 \leqq n(\varepsilon)$ we have

$$\sum_{(y_1, \ldots, y_{i+1}) \in Y^{i+1}} \bar{\nu}(y_1, \ldots, y_{i+m+1}) = \sum \nu(y_1, \ldots, y_{i+m+1}) = \nu(y_{i+2}, \ldots, y_{i+m+1})$$

by the stationarity of ν. If $i + m + 1 > n(\varepsilon)$, we use (1):

Only the first member of every product depends on y_1 and hence after summing we have

$$\sum_{y_1, \ldots, y_{i+1} \in Y^{i+1}} \bar{\nu}(y_1, \ldots, y_{i+1}, y_{i+2}, \ldots, y_{i+m+1}) =$$

$$= \sum_{y_2, \ldots, y_{i+1} \in Y^i} \left(\sum_{y_1 \in Y} \nu(y_1, \ldots, y_{n(\varepsilon)+1}) \right) \cdot \frac{\nu(y_2, \ldots, y_{n(\varepsilon)+2})}{(y_2, \ldots, y_{n(\varepsilon)+1})} \cdot \frac{\nu(y_h, \ldots, y_{i+m+1})}{\nu(y_h, \ldots, y_{i+m})} = W,$$

where $h = i + m + 1 - n(\varepsilon)$. The inner sum equals $\nu(E^{2,\ldots,n(\varepsilon)+1}_{y_2,\ldots,y_{n(\varepsilon)+1}})$.
According to the stationarity of ν we may write

$$W = \sum_{y_2,\ldots,y_{i+1} \in Y^i} \bar{\nu}(y_2, \ldots, y_{i+m+1}) = \nu(E^{i+1,\ldots,i+m}_{y_{i+2},\ldots,y_{i+m+1}})$$

and the second step of the induction is finished. Hence $\bar{\mathfrak{X}}$ is a stationary source.

Now we have to calculate $\varrho(\mathfrak{X}, \bar{\mathfrak{X}})$. By definition for $m \leqq n(\varepsilon)$ we have directly $\bar{\nu}(y_1, \ldots, y_m) = \nu(y_1, \ldots, y_m)$. If $m > n(\varepsilon)$ and $y_1, \ldots, y_m \in Y^m$ then $0 \leqq \nu(y_1, \ldots, y_m) \leqq \nu(y_1, \ldots, y_{n(\varepsilon)}) < \varepsilon$, $0 \leqq \bar{\nu}(y_1, \ldots, y_m) < \varepsilon$, which implies

$$|\nu(y_1, \ldots y_m) - \bar{\nu}(y_1, \ldots y_m)| < \varepsilon$$

and $\varrho(\mathfrak{X}, \bar{\mathfrak{X}}) < \varepsilon$. This completes the proof of Lemma 2.

We note that the probability distribution π

$$\bar{\pi}(y_1, \ldots, y_m) = \begin{cases} 0 \text{ for } m < n(\varepsilon), \\ \nu(y_1, \ldots, y_m) \quad \text{for } m = n(\varepsilon), \end{cases}$$

represents the stationary vector of the Markov chain (Z, π).

3. NON REGULAR SOURCES

The assertion of Lemma 2 can be reformulated in the following way: The set of all regular stationary a-generators is dense in the set of all regular stationary generators. Naturally, the question arises whether we can omit the words „regular" in the last statement. In the sequel we shall discuss this question.

Lemma 3. *To every stationary source $\mathfrak{X} = (Y^N, \mathbf{Y}^N, \nu)$ and every $\varepsilon > 0$ there exist a non-negative integer h, periodically degenerated sources $(Y^N, \mathbf{Y}^N, \nu)$, ..., $(Y^N, \mathbf{Y}^N, \nu_h)$ and a regular source $(Y^N, \mathbf{Y}^N, \nu_0)$ such that if we put $\nu' = \sum_{i=0}^{h} k_i \nu_i$* and $\mathfrak{X}' = (Y^N, \mathbf{Y}^N, \nu')$, *we have $\varrho(\mathfrak{X}, \mathfrak{X}') < \varepsilon$, (i.e. every stationary source can be approximated with the given accuracy by a „convex combination" of a regular and periodically degenerated sources).*

Proof. Let us denote $Y_0 = \{y \in Y^N : \nu(y) > 0\}$. Y_0 is finite or denumerable and measurable. $\nu(Y_0) \in < 0; 1 >$ and owing to the stationarity of $\mathfrak{X}$, $\nu(Y^k \times Y_0) = \nu(Y_0)$, for every $k \in N$. The set $Y^k \times Y_0$ is finite or denumerable and its measure is determined by its elements of positive measure. Therefore $Y_0 \subset Y^k \times Y_0$, which suggests that Y_0 has some periodic property. As a matter of fact, if $y \in Y_0$ and $F_k = Y^k \times \{y\}$, then evidently $\nu(F_k) = \nu(y) > 0$ for every $k \in N$, which implies that there exist j, $k \in N$, $j < k$ such that $F_j \cap F_k$ is a non empty set. Let $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots) \in F_j \cap F_k$, then for every $i \in N$ we have subsequently $\bar{y}_{k+i} = y_i, \bar{y}_{j+i} = y_i, \bar{y}_{j+(k-j)-i} = y_{i+(k-j)}$ and $y_i = y_{i+(k-j)}$. Thus Y_0 contains only periodical sequences.

It is evident that there exists a finite set $Y' \subset Y_0$ such that $\nu(Y_0) < \nu(Y') + \varepsilon\nu(Y^N - Y_0)$ if $\nu(Y^N - Y_0) \neq 0$ and $\nu(Y_0) < \nu(Y') + \varepsilon$ if $\nu(Y^N - Y_0) = 0$.

Let us denote $Y' = \{y(1), \ldots, y(h)\}$ and put for every $E \in \mathbf{Y}^N$

$$\nu_0(E) = \begin{cases} \dfrac{\nu(E \cap (Y^N - Y_0))}{\nu(Y^N - Y_0)} & \text{if } \nu(Y^N - Y_0) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$\nu_i(E) = \varkappa_E(y(i))$ for every $i \in N$. [$\varkappa_E$ is the characteristic function of the set E].

Further, let $k_0 = \nu(\mathbf{Y}^N - Y')$, $k_i = \nu(\{y(i)\})$.

Obviously, $\nu' = \sum_{i=0}^{h} k_i\nu_i$ is a probability measure which satisfies the lemma (note that ν_0 may be a trivial measure).

Theorem 1. *Let $\mathfrak{X} = (Y^N, \mathbf{Y}^N, \nu)$ be a stationary source and $\varepsilon > 0$. Then there exists a stationary a-generator $\bar{\mathfrak{X}} = (Y^N, \mathbf{Y}^N, \bar{\nu})$ such that $\varrho(\mathfrak{X}, \bar{\mathfrak{X}}) < \varepsilon$. If $\mathfrak{X}$ is regular, then $\bar{\mathfrak{X}}$ can be chosen also regular.*

Proof. To a given $\varepsilon/2$ there exists a $\mathfrak{X}' = (Y^N, \mathbf{Y}^N, \nu')$ with the properties of Lemma 3 (ε replaced by $\varepsilon/2$). Our proof will be completed if we find an a-generator T such that $\varrho(\mathfrak{X}', \mathfrak{X}) < \varepsilon/2$. We shall suppose $\nu(Y^N - Y_0) \neq 0$ (otherwise the proof can be simply modified). To the source $\mathfrak{X}_0 = (Y^N, \mathbf{Y}^N, \nu_0)$, where ν_0 is from (2), there exists an a-generator $\bar{\mathfrak{X}}_0 = (Y^N, \mathbf{Y}^N, \bar{\nu}_0)$ such that $\varrho(\mathfrak{X}, \mathfrak{X}_0) < \varepsilon/2$.

Let $\bar{\mathfrak{X}}_0$ be a composition of $\mathfrak{S}_0 = (Z_0^N, \mathbf{Z}_0^N, \pi_0)$ and some mapping γ_0 at the origin 2. The set of states Z_0 will be a subset of the set Z which we have to construct. The ing elements of Z will consist of states for $\nu_1, \ldots, \nu_h$ from (2).

Let $\nu_i(y(i)) = 1$. $y(i)$ is periodic, i.e. there exists $p(i) \in N$ such that j-th and $(j + p(i))$—th components of $y(i)$ are equal for every $j \in N$. Such sequences can be generated by „Markov“ sources with transition probabilities 0 or 1. Thus the corresponding a-generator $\bar{\mathfrak{X}}_i$ — the composition of $(Z_i^N, \mathbf{Z}_i^N, \pi_i)$ and the mapping γ_i — can be chosen in the form: $Z_i = \{z^i(1), \ldots, z^i(p(i))\}$, $\pi_i(z^i(2)/z^i(1)) = 1, \ldots, \pi_i(z^i(p(i))/z^i(p(i) - 1)) = 1$, $\pi_i(z^i(1)/z^i(p(i))) = 1$, the other transitions having probabilities 0; $\gamma_i(z^i(j)) = y(i)_j$ for $j = 1, \ldots, p(i)$.

Now we denote $Z = \{z\} \cup \bigcup_{i=0}^{h} Z_i$, where $z \notin \bigcup_{i=0}^{h} Z_i$ is a supplementary state (which will be an initial state).

Let $z_{i0} = z^i(p(i))$ be the initial state of Z_i. Let the origin be 2, (for $i = 1, \ldots, h$).

Put $\pi(z_{i0}/z) = \begin{cases} \nu(Y^N - Y') & \text{for } i = 0, \\ \nu(y(i)) & \text{otherwise.} \end{cases}$

Other transitions from z have the probability 0. If for some $i = 0, \ldots, h$ we have $z' \in Z_i$, then the transitions from z' are possible only into z_i with the probabilities $\pi_i(./z')$.

We have just defined a Markov source $\mathfrak{S} = (Z^N, \mathbf{Z}^N, \pi)$. Define the mapping γ by the requirement: It is

1. arbitrary on z,
2. equal to γ_i on Z_i.

Let $\mathfrak{T}$ be a composition of $\mathfrak{S}$ and γ at the origin 3. It is evident that $\bar{\nu} = \nu'$ on Y' and $| \bar{\nu} - \nu' | < \varepsilon/2$ on $(Y^N - Y')$, which completes the proof.

4. STOCHASTIC TRANSFORMERS

Throughout this section we suppose that there is given a stationary independent source $\mathfrak{U} = (X^N, \mathbf{X}^N, \mu)$ such that all $x \in X$ have equal probabilities. It is a trivial consequence of the results of [1], [2], [3] that for a given Markov source $\mathfrak{S} = (Z^N, \mathbf{Z}^N, \pi)$ there exists the g.s.m. $\mathscr{A}$ such that by the composition of $\mathfrak{U}$ and $\mathscr{A}$, i.e. by letting the input tape of $\mathscr{A}$ be the output tape of $\mathfrak{S}$, we can obtain a Markov source $\bar{\mathfrak{S}} = (Z^N, \mathbf{Z}^N, \pi)$ with the transition probabilities different from those of $\mathfrak{S}$ by less than a given $\varepsilon > 0$. In other words $\sigma(\mathfrak{S}, \mathfrak{S}) = \sup_{z_1 z_2 \in Z} |\pi(z_1/z_2) - \bar{\pi}(z_1/z_2)| < \varepsilon$.

Let γ be a mapping of Z into Y and let $\mathfrak{T}$, $\bar{\mathfrak{T}}$ be the compositions of $\mathfrak{S}$, γ, resp. $\bar{\mathfrak{S}}$, γ. Let $\mathfrak{S}$ be the source of the form from the proof of Theorem 1. What can we say about $\varrho(\mathfrak{T}, \bar{\mathfrak{T}})$?

Lemma 4. *Let $\mathfrak{S} = (Z^N, \mathbf{Z}^N, \pi)$, $\bar{\mathfrak{S}} = (Z^N, \mathbf{Z}^N, \pi)$ be Markov stationary sources. Let $\mathfrak{T}$, resp. $\bar{\mathfrak{T}}$ be the composition of $\mathfrak{S}$ resp. $\mathfrak{S}$ and a given mapping γ. Then to an arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\sigma(\mathfrak{S}, \bar{\mathfrak{S}}) < \delta$, then $\varrho(\mathfrak{T}, \bar{\mathfrak{T}}) < \varepsilon$.*

Proof. The assertion of Lemma 4 follows form the fact that the stationary vector of a Markov source is a continuous function of the transition probabilities.

We can now formulate a theorem on stochastic transformers which is a generalisation of those in [1], [2], [3].

Theorem 2. *Let $\mathfrak{U} = (X^N, \mathbf{X}^N, \mu)$ be a stationary independent source with equal probabilities of all letters from X and the number of elements in X greater than 1. Let $\mathfrak{T} = (Y^N, \mathbf{Y}^N, \nu)$ be a stationary source and let $\varepsilon > 0$. Then there exists a g.s.m. $\mathscr{A} = (A, X, Y, f, g, a_0)$ such that if $\tilde{\mathfrak{T}}$ is the composition of $\mathfrak{U}$ and $\mathscr{A}$, then $\varrho(\mathfrak{T}, \tilde{\mathfrak{T}}) < \varepsilon$.*

Proof. According to Theorem 1, there exists a stationary a-generator $\bar{\mathfrak{T}}$ such that $\varrho(\mathfrak{T}, \bar{\mathfrak{T}}) < \varepsilon/2$. Let $\bar{\mathfrak{T}}$ be the composition of $\bar{\mathfrak{S}}$ and γ. It follows from Lemma 4 that to $\varepsilon/2 > 0$ there exists a $\delta > 0$ such that if $\sigma(\mathfrak{S}', \bar{\mathfrak{S}}) < \delta$, then $\varrho(\mathfrak{T}', \bar{\mathfrak{T}}) < \varepsilon/2$, where $\mathfrak{T}'$ is the composition of $\mathfrak{S}'$ and γ. As we have mentioned above, for the Markov source $\bar{\mathfrak{S}}$ there exists a g.s.m. $\mathscr{A}'$ such that the composition $\mathfrak{S}'$ of $\mathfrak{U}$ and $\mathscr{A}'$ has the property that $\sigma(\mathfrak{S}', \mathfrak{S}) < \delta$. It is obvious that the mapping γ is also realisable by some g.s.m. $\mathscr{A}''$. Let A be a composition of A$'$ and $\mathscr{A}''$ and $\tilde{\mathfrak{T}}$ the composition of $\mathfrak{U}$ and $\mathscr{A}$. Then $\varrho(\tilde{\mathfrak{T}}, \mathfrak{T}) \leqq \leqq \varrho(\tilde{\mathfrak{T}}, \bar{\mathfrak{T}}) + \varrho(\bar{\mathfrak{T}}, \mathfrak{T}) < \varepsilon$ which completes the proof.

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