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ON (H, K) – DECOMPOSITIONS OF A COMPLETE GRAPH

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In paper [4] the concept of the decomposition of a complete undirected graph according to a given group was introduced, but only its special case, the simple decomposition, was investigated. In [5] this concept was enlarged also for directed graphs. In the present paper we shall study these decompositions generally, but we shall again consider only Abelian groups.

The decomposition of a complete (directed or undirected) graph n with n vertices according to a group H (which is a subgroup of the group of automorphisms of the graph $\langle n \rangle$) was defined in [4] and [5] as a decomposition $\mathscr{R}$ of the graph $\langle n \rangle$ into pairwise edge-disjoint subgraphs, each of which contains all vertices of the graph $\langle n \rangle$ and whose union is the graph $\langle n \rangle$, while to any element $\alpha \in H$ a graph $G(\alpha)$ of the decomposition $\mathscr{R}$ is assigned (by a one-to-one or many-to-one manner) so that if α, β are two elements of H, the image of the graph $G(\alpha)$ in the mapping β is the graph $G(\beta\,\alpha)$. If the assigning of the graphs $G(\alpha)$ to elements α is one-to-one, we have the so-called simple decomposition, which was investigated in [4] and [5]. Now we shall study the general case, when for $\alpha \neq \beta$ there may be $G(\alpha) = G(\beta)$.

Lemma 1. *Let a decomposition $\mathscr{R}$ of a graph $\langle n \rangle$ according to a group H bet given. All elements $\alpha \in H$, for which $G(\alpha) = G(\varepsilon)$ holds, where ε is the unit elemens of the group H, form a subgroup of the group H.*

Proof. Let K be the set of the elements α of H, for which $G(\alpha) = G(\varepsilon)$. Let $\alpha \in K$, $\beta \in K$. Then $G(\alpha\beta) = \alpha\,(G(\beta)) = \alpha\,(G(\varepsilon)) = G(\alpha\varepsilon) = G(\alpha) = G(\varepsilon)$ and therefore $\alpha\beta \in K$. Further evidently $\varepsilon \in K$. Now let α^{-1} be the inverse element for $\alpha \in K$. We have $G(\alpha^{-1}) = G(\alpha^{-1}\varepsilon) = \alpha^{-1}(G(\varepsilon)) = \alpha^{-1}(G(\alpha)) = G(\alpha^{-1}\alpha) = G(\varepsilon)$ and therefore also $\alpha^{-1} \in K$. Herewith it is proved that K is a subgroup of the group H.

Lemma 2. *Let a decomposition $\mathscr{R}$ of a graph $\langle n \rangle$ according to an Abelian group H be given. Let $\gamma \in H$. The set of the elements $\beta \in H$ such that $G(\beta) = G(\gamma)$ is the class $\gamma\, K$ of the decomposition of the group H according to the subgroup K consisting of the elements α such that $G(\alpha) = G(\varepsilon)$.*

Proof. Let $G(\beta) = G(\gamma)$. Then $G(\varepsilon) = \gamma^{-1}(G(\gamma)) = \gamma^{-1}(G(\beta)) = G(\gamma^{-1}\beta)$ and therefore $\gamma^{-1}\beta \in K$, thus $\beta \in \gamma K$. On the other hand, if $\beta \in \gamma K$, this means that $\beta = \gamma\alpha$, where $\alpha \in K$. Then $G(\beta) = G(\gamma\alpha) = \gamma(G(\alpha)) = \gamma(G(\varepsilon)) = G(\gamma)$.

Now we shall express a definition.

Definition. Let H be a subgroup of the group of automorphisms of the complete graph $\langle n \rangle$ with n vertices (directed or undirected), let K be its subgroup. By the (H, K)-decomposition of the graph $\langle n \rangle$ we shall mean a decomposition $\mathscr{R}$ of this graph into pairwise edge-disjoint subgraphs, each of which contains all vertices of the graph $\langle n \rangle$ and whose union is the graph $\langle n \rangle$, while the following condition is satisfied: to each element α of the group H a graph $G(\alpha)$ of the decomposition $\mathscr{R}$ is assigned so that if α, β are two elements of the group H, the image of the graph $G(\alpha)$ in the mapping β is the graph $G(\beta\alpha)$; each graph of the decomposition $\mathscr{R}$ can be expressed as $G(\alpha)$ for some $\alpha \in H$ and K is the set of exactly all elements γ of H such that $G(\gamma) = G(\varepsilon)$, where ε is the unit element of the group H.

We shall express an existence theorem.

Theorem 1. *Let $\vec{K}_n$ be the complete directed graph with n vertices; let H be an Abelian group; let K be its subgroup. Let K be the direct product of the primary cyclic subgroups $K_1, \ldots, K_k$. The necessary and sufficient condition of the existence of a (H, K)-decomposition of the graph $\vec{K}_n$ is that the quotient $o(H)/o(K)$ should be a divisor either of the number n, or of the number $n - 1$, and that in the first case there should be $no(K)/o(H) \geqq \sum_{i=1}^{k} o(K_i)$, in the second case $(n - 1)o(K)/o(H) \geqq \sum_{i-1}^{k} o(K_i)$, where $o(H), o(K), o(K_1), \ldots, o(K_k)$ are consequently the orders of the groups $H, K, K_1, \ldots, K_n$.*

Remark. We must emphasize that we study only such groups H which are subgroups of the group of automorphisms of the graph $\langle n \rangle$. Otherwise it would be possible to construct (H, K)-decompositions for groups H of arbitrary high orders.

Proof. At first we shall prove the necessity of the condition. Let an (H, K)-decomposition $\mathscr{R}$ of the graph $\vec{K}_n$ exist. Lemma 2 implies that the decomposition $\mathscr{R}$ consists exactly of $o(H)/o(K)$ different graphs (because this is the index of the subgroup K in the group H). Let some vertex u of the graph $\vec{K}_n$ be fixed in some mapping $\alpha \in H \dot{-} K$ and let a mapping $\beta \in H$ exist, in which the vertex u is not fixed. Let h be the edge joining the vertices u and $\beta(u)$. Then $\alpha(h) = \alpha(u)\, \alpha\beta(u) = \alpha(u)\, \beta\alpha(u) = u\beta(u) = h$. Let the edge h belong to a graph $G(\gamma)$ of the decomposition $\mathscr{R}$, where $\gamma \in H$. Then it must belong at the same time to the graph $G(\alpha\gamma)$ and therefore, as any two different graphs of the decomposition $\mathscr{R}$ are edge-disjoint, it must be $G(\gamma) = G(\alpha\gamma)$. This implies $G(\alpha) = \gamma^{-1}(G(\alpha\gamma)) = \gamma^{-1}(G(\gamma)) = G(\gamma^{-1}\gamma) = G(\varepsilon)$ and thus $\alpha \in K$, which is a contradiction with the assumption that $\alpha \in H \dot{-} K$. Therefore if the vertex u is fixed in some mapping of $H \dot{-} K$, it is fixed in all mappings of H. But such a vertex can be at most one; if there existed two different ones, the edge joining them would also be fixed and would belong to all graphs of the decomposition $\mathscr{R}$ at the same time, which is impossible.

Thus let u be a vertex of the graph $\vec{K}_n$ and let H_u be the subgroup of the group H consisting of all mappings of H, in which the vertex u is fixed; according to the above mentioned there is $H_u \subset K$ for all vertices u of the graph $\vec{K}_n$ except for at most one vertex fixed in all mappings of H. The number of pairwise different images of the vertex u (which is not fixed in all mappings of H) is therefore equal to the index of the group H_u in the group H (because two mappings of H transform u in the same vertex evidently if and only if they both belong to the same class according to H_u). The index of the group H_u in H will be an integral multiple of the index of the group K in H, because $H_u \subset K$. Thus the number of pairwise different images of the vertex u in mappings of H is an integral multiple of the number $o(H)/o(K)$.

Now decompose the set U of the vertices of the graph $\vec{K}_n$ into classes A_1, ..., A_r such that two vertices belong to the same class if and only if one of them is the image of another in some mapping of H. According to the above proved, the number of vertices of any of the classes $A_1, \ldots, A_r$ is a multiple of $o(H)/o(K)$, except for at most one class formed by the vertex fixed in all mappings of H. The number of the vertices of the graph $\vec{K}_n$ is the sum of numbers of vertices of all the classes $A_1, \ldots, A_r$; if none of them consists of one element only, then n is a multiple of the number $o(H)/o(K)$, in the reverse case $n - 1$ is that multiple. Let $1 \leqq i \leqq k$ and take the group K_i. Among the vertices of the graph K_n there must exist at least one vertex u such that for $\alpha \in K_i$, $\beta \in K_i$, $\alpha \neq \beta$, there is $\alpha(u) \neq \beta(u)$. We shall prove this by a contradiction. It is well-known that the system of all subgroups of a finite cyclic primary group is totally ordered in respect to the inclusion. Thus to any vertex u let two elements $\alpha \in K_i$, $\beta \in K_i$ exist such that $\alpha \neq \beta$, $\alpha(u) = \beta(u)$. Then $u = \alpha^{-1}\alpha(u) = \alpha^{-1}\beta(u)$. In this case the images of the vertex u in the whole subgroup of the group K_i generated by the element $\alpha^{-1}\beta$ are identical. This means that to any vertex u there exists a subgroup $K_i(u)$ of the group K_i with more than one element such that $\gamma(u) = u$ for all elements γ of that group. Let $K_i' = \bigcap_{u \in U} K_i(u)$. The total ordering and finiteness of the system of subgroups of the group K_i implies that K_i' is the minimal (with respect to the ordering according to the inclusion) of the groups $K_i(u)$, therefore it is a group with more than one element. Let δ be an element of this group different from the unit element. There is $\delta(u) = u$ for all vertices of the graph $\vec{K}_n$. But as H (and therefore also K_i) is a subgroup of the group of automorphisms of the graph $\vec{K}_n$, this implies that $\delta = \varepsilon$, which is a contradiction. Thus to each of the groups K_i, $1 \leqq i \leqq k$, there exists a vertex, whose images in different mappings of this group are different and which has the number of images equal at least to $o(K_i)o(H)/o(K)$ (because it can be fixed at most in the mappings of $K \dot{-} K_i$ and in the identical mapping). If some vertex has the above mentioned property with respect to some groups K_i at the same time, then the number of its vertices is equal to at least the product of their orders. Let a_j be the number of vertices of the class A_j. As for positive integers a, b greater than one, $a + b \leqq ab$ holds, then $n = \sum_{j=1}^{r} a_j \geqq$ $\geqq \sum_{i=1}^{k} o(K_i)o(H)/o(K)$, that is, $no(K)/o(H) \geqq \sum_{i=1}^{k} o(K_i)$ in the case when there does not exist any vertex fixed in all mappings of H and $n \geqq 1 + \sum_{i=1}^{k} o(K_i)o(H)/$ $/o(K)$, that is, $(n - 1)o(K)/o(H) \geqq \sum_{i=1}^{k} o(K_i)$ in the reverse case.

Now let the condition be satisfied. At first assume that the first case occurred. From the set U choose a system of k subsets $B_1, \ldots, B_k$ such that any two of them are disjoint and B_i contains $o(K_i)o(H)/o(K)$ vertices, where $1 \leqq i \leqq k$. Let $\bar{K}_i$ be the direct product of all groups K_j, where $1 \leqq j \leqq k, j \neq i$ (in the case $k = 1$ we have $\bar{K}_i = \{\varepsilon\}$), further let $K_i^* \simeq H/K_i$. The group K_i^* has evidently the order $o(K_i)o(H)/o(K)$. By a one-to-one manner assign to each element $\alpha^* \in K_i^*$ a vertex $u_i(\alpha^*) \in B_i$. If $\beta \in H$, let β_i^* be the image of the element β in the natural homomorphism of the group H onto the factorgroup K_i^*. Then define $\beta(u_i(\alpha^*)) = u_i(\beta^*\alpha^*)$. Further decompose the set $U - \bigcup_{i=1}^{k} B_i$ into the sets $C_1, \ldots, C_l$, which are again pairwise disjoint and each of which contains $o(H)/o(K)$ vertices (if $U - \bigcup_{i=1}^{k} B_i \neq \emptyset$). Let $K^* \simeq H/K$. Assign again to any element $\gamma^* \in K^*$ by a one-to-one manner a vertex $v_i(\gamma^*) \in C_i$. If $\beta \in H$, let β^* be the image of the element β in the natural homomorphism of the group H onto the factorgroup K^*. Then we define $\beta(v_i(\gamma^*)) = v_i(\beta^*\gamma^*)$. Now choose an arbitrary edge h of the graph K_n and put it into $G(\varepsilon)$; further for any $\alpha \in H$ put the edge $\alpha(h)$ into $G(\alpha)$. Then we choose again an edge which was not put yet into any of the graphs $G(\alpha)$ and do the same with it. Thus we proceed until each edge is put into some of the graphs $G(\alpha)$. Now let an edge k belong at the same time to the graphs $G(\varphi)$ and $G(\psi)$; we shall prove that then both φ and ψ belong to the same class of the group H according to the subgroup K. The edge k is evidently the image of some chosen edges h_1, h_2 which were put into $G(\varepsilon)$ in the mappings φ and ψ, thus $k = \varphi(h_1) = \psi(h_2)$. Then $h_2 = \psi^{-1}\varphi(h_1)$ and h_2 belongs to both the graphs $G(\varepsilon)$ and $G(\psi^{-1}\varphi)$. Let $h_1 \neq h_2$. Assume without the loss of generality that h_1 was chosen in our procedure before h_2. Then, having completed the procedure with h_1 we could not choose h_2, because it was still put into $G(\psi^{-1}\varphi)$. This is a contradiction. If $h_1 = h_2$, then $\varphi(h_1) = \psi(h_1)$. If w_1 is the beginning vertex, w_2 the end vertex of the edge h_1, then $\varphi(w_1) = \psi(w_1)$, $\varphi(w_2) = \psi(w_2)$ and therefore $\varphi^{-1}\psi(w_1) = w_1$, $\varphi^{-1}\psi(w_2) = w_2$. If $w_1 \in B_i$, $1 \leqq i \leqq k$, then $w_1 = u_i(\alpha^*)$ for some $\alpha^* \in K_i^*$. Therefore $u_i(\alpha^*) = \varphi^{-1}\psi\,(u(\alpha^*)) = u(\varphi^{*-1}\psi^*\alpha^*)$, where φ^*, ψ^* are images of the elements φ, ψ in the natural homomorphism of the group H onto K_i^*. But as the assigning of vertices of B_i to the elements of K_i^* is one-to-one, there is $\alpha^* = \varphi^{*-1}\psi^*\alpha^*$, therefore $\varphi^* = \psi^*$ and the elements φ, ψ belong to the same class of the group H according to K_i, therefore also according to K. Analogously in the case $w_1 \in C_i$, $1 \leqq i \leqq l$. Thus we may identify any two graphs $G(\varphi)$, $G(\psi)$, where $\varphi^{-1}\psi \in K$ and we obtain thus the wanted decomposition.

Analogously in the second case; there we must add the vertex fixed in all mappings of H.

We shall prove another theorem which shows the connection between (H, K)-decompositions and decompositions of a complete graph into two isomorphic subgraphs.

Theorem 2. *A decomposition of a complete graph (directed or undirected) into two isomorphic edge-disjoint subgraphs is a (H, K)-decomposition, where H is an Abelian group of an even or infinite order and K is its subgroup with the index 2.*

Proof. Let the graph $\langle n \rangle$ be decomposed into two isomorphic edge-disjoint subgraphs G and $\bar{G}$ let φ be an isomorphic mapping of the graph G and $\bar{G}$. First consider a finite graph. Each cycle of the permutation p induced on the set U of the vertices of the graph $\langle n \rangle$ by the mapping φ, except for at most one cycle formed by a fixed vertex, has an even number of vertices, in undirected graphs even divisible by four [2, 3, 5]. Let q be the least common multiple of numbers of vertices of the cycles of the permutation p induced on the vertex set U of the graph $\langle n \rangle$ by the mapping φ. Take a cyclic group H of the order q. Let its generator be α. Define $\alpha^r(u) = \varphi^r(u)$, where r is an arbitrary integer and $u \in U$. As for q we have chosen the least common multiple of numbers of vertices of the cycles, evidently for each pair of elements of H there exists a vertex whose images in these mappings are different, because if there were $\alpha^m(u) = \alpha^n(u)$ for $0 < m < n \leqq q$ and for all vertices $u \in U$, there would be $\alpha^{n-m}(u) = u$ for all vertices $u \in U$ and therefore $\alpha^{n-m} = \varphi^{n-m}$ would be an identical mapping, which would imply that $n - m$ is a multiple of q, but this contradicts the assumption that $0 < m < n \leqq q$. Thus the group H is really a subgroup of the group of automorphisms of the graph $\langle n \rangle$. Let K be a subgroup of the group H generated by the element α^2.

Then the decomposition of the graph $\langle n \rangle$ into the graphs G and $\bar{G}$ is evidently a (H, K)-decomposition, while $G = G(\varepsilon)$, $\bar{G} = G(\alpha)$. In the case of an infinite graph it may not be possible to determine the least common multiple of numbers of vertices of the cycles of the permutation p, if some cycle contains infinitely many elements or if the numbers of vertices of the cycles are finite, but they have no common multiples (which can occur at infinitely many numbers). Then we choose for H the infinite cyclic group. The further procedure of the proof is analogous.

The decompositions of a complete graph into two isomorphic edge-disjoint subgraphs are studied in papers [1, 2, 3, 5].

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