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MIXTURE SETS AND CONVEX SETS

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In [1, 2] the concept of a mixture set was introduced and used for statistical purposes. Mixture sets keep those properties of convex sets in linear spaces which are important for „statistical mixing" of elements. The aim of this paper is to show more exactly the relation between mixture sets and convex sets.

Definition. *A mixture set* (R, Φ) *is a set* $R \neq \emptyset$ *and an operation* $\Phi_\alpha(r_1, r_2)$ *associating an element of* R *with each* $\alpha \in \langle 0, 1\rangle$ *and each ordered pair* $(r_1, r_2) \in R \times R$ *such that if* $\alpha, \beta \in \langle 0, 1\rangle$ *and* $r_1, r_2 \in R$ *then*

M1: $\Phi_\alpha(r_1, r_2) = \Phi_{1-\alpha}(r_2, r_1)$,
M2: $\Phi_1(r_1, r_2) = r_1$,
M3: $\Phi_\alpha [\Phi_\beta(r_1, r_2), r_2] = \Phi_{\alpha\beta}(r_1, r_2)$.

Lemma 1. [2] *In a mixture set* (R, Φ) *for each* $r_1, r_2 \in R$, $\alpha, \beta, \gamma \in \langle 0, 1\rangle$ *we have*

$$\Phi_\alpha(r_1, r_1) = r_1, \tag{1}$$

$$\Phi_\alpha [\Phi_\beta(r_1, r_2), \Phi_\gamma(r_1, r_2)] = \Phi_{\alpha\beta+(1-\alpha)\gamma}(r_1, r_2). \tag{2}$$

Proof. From M1 and M2 we obtain

$$r_1 = \Phi_1(r_1, r_1) = \Phi_0(r_1, r_1),$$

and from M3

$$r_1 = \Phi_{\alpha 0}(r_1, r_1) = \Phi_\alpha [\Phi_0(r_1, r_1), r_1] = \Phi_\alpha(r_1, r_1).$$

If $\gamma = 0$ or $\beta = 0$, then (2) follows directly from M3. Hence let us suppose that $0 < \beta \leq \gamma$ (If $0 < \gamma \leq \beta$, the proof is similar.) Using M3 we obtain

$$\Phi_\beta(r_1, r_2) = \Phi_{\beta/\gamma}[\Phi_\gamma(r_1, r_2), r_2]$$

and hence from M2 and M3

$$\Phi_\alpha[\Phi_\beta(r_1, r_2), \Phi_\gamma(r_1, r_2)] = \Phi_\alpha\{\Phi_{(1-\beta/\gamma)}[r_2, \Phi_\gamma(r_1, r_2)], \Phi_\gamma(r_1, r_2)\}$$

$$= \Phi_{\alpha(1-\beta/\gamma)} [r_2, \Phi_\gamma(r_1, r_2)] = \Phi_{1-\alpha(1-\beta/\gamma)}[\Phi_\gamma(r_1, r_2), r_2]$$

$$= \Phi_{\alpha\beta+(1-\alpha)\gamma}(r_1, r_2).$$

Every convex subset of any linear space (a covex set) is a mixture set if we take $\Phi_\alpha(r_1, r_2) = \alpha r_1 + (1-\alpha) r_2$. The reverse of this statement is not always true as it is shown in the examples which follow later.

Definition. *Two mixture sets (R, Φ) and (S, Ψ) are isomorphic if there is a one-to-one map π mapping R onto S such that $\pi[\Phi_\alpha(r_1, r_2)] = \Psi_\alpha[\pi(r_1), \pi(r_2)]$.*

Lemma 2. *Let (R, Φ) be a mixture set isomorphic to a convex set. If $\alpha, \beta \in \langle 0, 1\rangle$ and $r_1, r_2, r_3 \in R$, then*

$$(3) \qquad (\alpha = \beta) \vee (r_1 = r_2) \Leftrightarrow \Phi_\alpha(r_1, r_2) = \Phi_\beta(r_1, r_2)$$

and

M4: $(\alpha + \beta \leqq 1) \wedge (\alpha < 1) \wedge (\beta < 1) \Rightarrow$

$$\Phi_\alpha[r_1, \Phi_{\beta/(1-\alpha)}(r_2, r_3)] = \Phi_\beta[r_2, \Phi_{\alpha/(1-\beta)}(r_1, r_3)].$$

Proof.

$$1. \quad \Phi_\alpha(r_1, r_2) = \Phi_\beta(r_1, r_2) \Leftrightarrow \alpha\pi(r_1) + (1-\alpha)\pi(r_2) = \beta\pi(r_1) + (1-\beta)\pi(r_2) \Leftrightarrow (\alpha - \beta)\pi(r_1) = (\alpha-\beta)\pi(r_2) \Leftrightarrow (\alpha = \beta) \vee (\pi(r_1) = \pi(r_2)).$$

$$2. \quad \Phi_\alpha[r_1, \Phi_{\beta/(1-\alpha)}(r_2, r_3)] = \pi^{-1}\{\alpha\pi(r_1) + (1-\alpha)[(\beta/(1-\alpha))\pi(r_2) + (1 - \beta/(1-\alpha))\pi(r_3)]\} = \pi^{-1}\{\beta\pi(r_2) + (1-\beta)[(\alpha/(1-\beta))\pi(r_1) + (1-\alpha/(1-\beta))\pi(r_3)]\} = \Phi_\beta[r_2, \Phi_{\alpha/(1-\beta)}(r_1, r_3)].$$

Definition. *A mixture set (R, Φ) is a topological mixture set if R is a Hausdorf space and $\Phi_\alpha(r_1, r_2)$ is continuous with respect to the product topology on $\langle 0, 1\rangle \times R \times R$.*

Example 1. Let us put $R = \{-1\} \cup \langle 0, 1\rangle$ a set of real numbers. Let us define $\Phi_\alpha(r_1, r_2) = \alpha r_1 + (1-\alpha) r_2$ if $r_1, r_2 \in \langle 0, 1\rangle$ and $\alpha \in \langle 0, 1\rangle$; $\Phi_\alpha(-1, r) = \Phi_\alpha(0, r)$ if $\alpha \in \langle 0, 1)$ and $r \neq -1$; $\Phi_\alpha(-1, r) = -1$ if $\alpha = 1$ or $r = -1$; $\Phi_\alpha(r, -1) = \Phi_{1-\alpha}(-1, r)$ for $\alpha \in \langle 0, 1\rangle$. (R, Φ) is a mixture set where M4 is valid, but $\Phi_{1/2}(-1, 1) = \Phi_{1/2}(0, 1)$, hence (R, Φ) is not (isomorphic to) a convex set.

Example 2. (R, ϱ) is a metric space with the metric ϱ such that for every $r_1, r_2 \in R$ and every $\alpha \in \langle 0, 1\rangle$ a unique solution r of the equations

$$(4) \qquad \varrho(r_1, r) = (1-\alpha)\varrho(r_1, r_2),$$

$$\varrho(r_2, r) = \alpha\varrho(r_1, r_2)$$

does exist. We define $\Phi_\alpha(r_1, r_2) = r$. (R, Φ), is a mixture set. M1 and M2 are evident and we shall prove only M3. Denote $r_3 = \Phi_\alpha(r_1, r_2)$, $r_4 = \Phi_\beta(r_3, r_2)$. From (4) and the properties of a metric we obtain

$$\varrho(r_4, r_2) = \beta\varrho(r_3, r_2) = \alpha\beta\varrho(r_1, r_2),$$

$$\varrho(r_4, r_1) \geqq \varrho(r_1, r_2) - \varrho(r_4, r_2) = (1 - \alpha\beta)\varrho(r_1, r_2),$$

and

$$\varrho(r_4, r_1) \leqq \varrho(r_4, r_3) + \varrho(r_3, r_1) = (1 - \beta)\varrho(r_2, r_3) + (1 - \alpha)\varrho(r_1, r_2) =$$
$$= (1 - \alpha\beta)\varrho(r_1, r_2),$$

hence $r_4 = \Phi_{\alpha\beta}(r_1, r_2)$.

As a special case let us consider a subset R of the unit sphere in the 3-dimensional Euclidean space with the metric $\varrho(r_1, r_2)$ equal to the arc length between r_1 and r_2, measured on the great circle through r_1 and r_2. We take $R = \{r : r = (\varphi, \vartheta), \varphi \in \langle 0, \pi/2\rangle, \vartheta \in 0, \pi/2\rangle\}$, where φ, ϑ are usual spherical coordinates. (R, Φ) is a topological (metric) mixture set, but it is not (isomorphic to) a convex set since M4 is not valid. To show that, it is sufficient to take $r_1 = (\varphi_1 = 0, \vartheta_1 = \pi/2)$, $r_2 = (\varphi_2 = \pi/2, \vartheta_2 = \pi/2)$, $r_3 = (\varphi_3, \vartheta_3 = 0)$. If we denote $r_4 = \Phi_\alpha [r_1, \Phi_{\beta/(1-\alpha)} (r_2, r_3)]$ and $r_5 = \Phi_\beta [r_2, \Phi_{\alpha/(1-\beta)}(r_1, r_3)]$, we obtain by elementary computations

$$\cos^2 \vartheta_4 = [1 - (\tfrac{1}{2} - \sin^2 ((1 - \alpha)\pi/4))^2]/[1 + \mathrm{tg}^2 (\beta\pi/2(1 - \alpha))],$$

$$\cos^2 \vartheta_5 = [1 - (\tfrac{1}{2} - \sin^2 ((1 - \beta)\pi/4))^2]/[1 + \mathrm{tg}^2 (\alpha\pi/2(1 - \beta))].$$

If $\alpha = 4/7$ and $\beta = 1/7$ then $\alpha + \beta < 1$, $\mathrm{tg}^2 [\beta\pi/2(1 - \alpha)] = \mathrm{tg}^2 \pi/6 = 1/3$, $\mathrm{tg}^2 [\alpha\pi/2(1 - \beta)] = \mathrm{tg}^2 \pi/3=3$, $(1 - \alpha)/4 = 3/28$, $(1 - \beta)/4 = 3/14$. Therefore $\cos^2 \vartheta_4 = [1 - (\tfrac{1}{2} - \sin^2 (3\pi/28))^2]\, 3/4 > [1-(1/2)^2]3/4 > [1 - (\tfrac{1}{2} - \tfrac{1}{2})^2 1/4 > > [1 - \tfrac{1}{2} - \sin^2 (3\pi/14))^2]1/4 = \cos^2 \vartheta_5$.
Thus $r_4 \neq r_5$ and M4 is not true.

Lemma 3. *In a topological mixture set* (R, Φ) *the equivalence* (3) *is valid.*

Proof. The right-hand side of (3) follows directly from the left-hand side using (1).

Suppose now that $\alpha_1 \neq \alpha_2$ (say $\alpha_1 < \alpha_2$), $r_1 \neq r_2$ and $\Phi_{\alpha_1}(r_1, r_2) = \Phi_{\alpha_2}(r_1, r_2) = r$. If $\alpha_2 < 1$, we denote $\Delta = (\alpha_2 - \alpha_1)/(1 - \alpha_1)$. Evidently $\Delta \in (0, 1)$ and $\alpha_2 = (1 - \Delta)\alpha_1 + \Delta$. We define $\alpha_n = (1 - \Delta)\alpha_{n-1} + \Delta$ for $n = 3, 4, \ldots$ Then $\lim_{n\to\infty} (1 - \alpha_n) = (1 - \alpha_1) \lim_{n\to\infty} (1 - \Delta)^{n-1} = 0$, hence $\lim_{n\to\infty} \alpha_n = 1$. If $\Phi_{\alpha_i}(r_1, r_2) = r$ for $i = 1, 2, \ldots, n - 1$, then, according to (2),

$$\Phi_{\alpha_n}(r_1, r_2) = \Phi_{\Delta 1+(1-\Delta)\alpha_{n-1}}(r_1, r_2) = \Phi_\Delta[r_1, \Phi_{\alpha_{n-1}}(r_1, r_2)] =$$
$$= \Phi_\Delta[r_1, \Phi_{\alpha_{n-2}}(r_1, r_2)] = \Phi_{\alpha_{n-1}}(r_1, r_2) = r.$$

Thus $\Phi_{\alpha_n}(r_1, r_2) = r$ for $n = 1, 2, \ldots$, and from the continuity of Φ we obtain $r_1 = \Phi_1(r_1, r_2) = \lim_{n\to\infty} \Phi_{\alpha_n}(r_1, r_2) = r$.

Since $\Phi_{(1-\alpha_1)}(r_2, r_1) = \Phi_{(1-\alpha_2)}(r_2, r_1)$, we repeat the proof putting $\alpha_1' = 1 - \alpha_2 < 1 - \alpha_1 = \alpha_2'$, and obtain $r = \lim_{n\to\infty} \Phi_{\alpha_n}(r_2, r_1) = \Phi_1(r_2, r_1) = r_2$. Hence $r_1 = r = r_2$, which contradicts the assumption that $r_1 \neq r_2$.

Lemma 4. *Let (R, Φ) be a topological mixture set with the property* M4. *If $r_1, r_2, r_3 \in R$ and $0 < \alpha \leqq \beta \leqq 1$, then $\Phi_\alpha(r_2, r_1) = \Phi_\beta(r_3, r_1) \Rightarrow (\exists \gamma \in \in (0, 1\rangle)(r_3 = \Phi_\gamma(r_2, r_1))$.*

Proof. If $r_1 = r_2 = r_3$, the statement is trivial and we can exclude this case. Let

$$\bar{\beta} = \sup \{\beta' : (\exists \alpha' \in (0, \beta'\rangle)\, (\Phi_{\alpha'}\, (r_2, r_1) = \Phi_{\beta'}\, (r_3, r_1))\}.$$

Since Φ is continuous we have $\Phi_{\bar{\alpha}}(r_2, r_1) = \Phi_{\bar{\beta}}(r_3, r_1)$ for some $\bar{\alpha} \in \langle 0, \bar{\beta}\rangle$. If $\bar{\alpha} = 0$, then $r_1 = \Phi_{\bar{\alpha}}(r_2, r_1) = \Phi_{\bar{\beta}}(r_3, r_1)$, and according to Lemma 3 and from $\bar{\beta} > 0$ we obtain $r_3 = r_1$. Hence $\Phi_\alpha(r_2, r_1) = \Phi_\beta(r_3, r_1) = r_1$ and, since $\alpha > 0$, we have $r_2 = r_1$. Thus $\bar{\alpha} \in (0, \bar{\beta}\rangle$. Suppose first that $\bar{\beta} < 1$. The numbers $\delta = \bar{\alpha}(1 - \bar{\beta})/(1 - \bar{\alpha}\bar{\beta})$ and $\varepsilon = \bar{\beta}(1 - \bar{\alpha})/(1 - \bar{\alpha}\bar{\beta})$ are such that $\varepsilon, \delta \in (0, 1)$, $\bar{\alpha} = \delta/(1 - \varepsilon)$ and $\bar{\beta} = \varepsilon/(1 - \delta)$. From (2) and M4 it follows

$$\Phi_{\delta+(1-\delta)\delta/(1-\varepsilon)}(r_2, r_1) = \Phi_\delta[r_2, \Phi_{\delta/(1-\varepsilon)}(r_2, r_1)] = \Phi_\delta[r_2, \Phi_{\varepsilon/(1-\delta)}(r_3, r_1)] =$$

$$\Phi_\varepsilon[r_3, \Phi_{\delta/(1-\varepsilon)}(r_2, r_1)] = \Phi_\varepsilon[r_3, \Phi_{\varepsilon/(1-\delta)}(r_3, r_1)] = \Phi_{\varepsilon+(1-\varepsilon)\varepsilon/(1-\delta)}(r_3, r_1).$$

But $\varepsilon + (1 - \varepsilon)\varepsilon/(1 - \delta) = \bar{\beta} + (1 - \bar{\beta})\varepsilon > \bar{\beta}$. This is a contradiction to the definition of $\bar{\beta}$. Hence $\bar{\beta} = 1$ and $r_3 = \Phi_{\bar{\alpha}}(r_2, r_1)$.

Definition. *Let (R, Φ) be a mixture set with the property* M4. *For each $r_i \in R$, $\alpha_i \subset \langle 0, 1\rangle$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} \alpha_i = 1$, $n \geqq 2$ we define an element of R, $\sum_{i=1}^{n}{}' \alpha_i r_i$, such that*

$$\sum_{i=1,2}{}' \alpha_i r_i = \Phi_{\alpha_1}(r_1, r_2) \tag{5a}$$

$$\sum_{i=1}^{n}{}' \alpha_i r_i = \begin{cases} \Phi_{\alpha_n}[r_n, \sum_{i=1}^{n-1}{}' (\alpha_i/(1 - \alpha_n))r_i], & \text{if } \alpha_n < 1 \\ = r_n, & \text{if } \alpha_n = 1 \end{cases} \tag{5b}$$

for $n = 3, 4, \ldots$.

Lemma 5.

$$\sum_{k=1}^{n}{}' \alpha_{i_k} r_{i_k} = \sum_{i=1}^{n}{}' \alpha_i r_i, \tag{6}$$

where $(i_1, \ldots, i_n)$ is an arbitrary permutation of $(1, \ldots, n)$.

Proof. From M4 it follows that

$$\sum_{i=1}^{n}{}' \alpha_i r_i = \Phi_{\alpha_n}[r_n, \Phi_{\alpha_{n-1}/(1-\alpha_n)}(r_{n-1}, \sum_{i=1}^{n-2}{}' \beta_i r_i)] =$$

$$\Phi_{\alpha_{n-1}}[r_{n-1}, \Phi_{\alpha_n/(1-\alpha_{n-1})}(r_n, \sum_{i=1}^{n-2}{}' \beta_i r_i)] = \sum_{k=1}^{n}{}' \alpha_{i_k} r_{i_k},$$

where $i_1 = 1$, $i_2 = 2$, $\ldots$, $i_{n-2} = n-2$, $i_{n-1} = n$, $i_n = n-1$, and $\beta_i = \alpha_i/(1-\alpha_{n-1}-\alpha_n)$, $i = 1, \ldots, n-2$. Using this, (6) is proved by induction with respect to n.

Lemma 6. *Let be* $\alpha_i \in (0, 1)$, $r_i \in R$, $i = 1, \ldots, m$, $m \geqq 2$, $1 \leqq n < m$, $\sum_{i=1}^{n} \alpha_i = 1$, $\sum_{i=n+1}^{m} \alpha_i = 1$. $\beta \in (0, 1)$.

Then

$$\Phi_\beta \left(\sum_{i=1}^{m-n-1}{}' \alpha_i r_i, \sum_{i=m-n}^{m}{}' \alpha_i r_i \right) = \sum_{i=1}^{m}{}' \gamma_i r_i \tag{7}$$

where $\gamma_i = \alpha_i \beta$ *for* $i = 1, \ldots, m-n$; $\gamma_i = \alpha_i(1-\beta)$ *for* $i = m-n+1, \ldots, m$.

Proof. For $n = 1$ (7) follows directly from (5b). If (7) is valid for $n' < n$ then from (5b), (6), and M4 we obtain

$$\Phi_\beta \left(\sum_{i=1}^{m-n-1}{}' \alpha_i r_i, \sum_{i=m-n}^{m}{}' \alpha_i r_i \right) = \Phi_\beta \left[\sum_{i=1}^{m-n-1}{}' \alpha_i r_i, \Phi_{\alpha_{m-n}}(r_{m-n}, \sum_{i=m-n+1}^{m}{}' [\alpha_i/(1-\alpha_{m-n})] r_i) \right] =$$

$$\Phi_{(1-\beta)\alpha_{m-n}}[r_{m-n}, \Phi_{\beta/[1-(1-\beta)\alpha_{m-n}]} \left(\sum_{i=1}^{m-n-1}{}' \alpha_i r_i, \sum_{i=m-n+1}^{m}{}' [\alpha_i/(1-\alpha_{m-n})] r_i \right)]$$

$$= \Phi_{(1-\beta)\alpha_{m-n}}[r_{m-n}, \sum_{\substack{i=1 \\ i \neq m-n}}^{m}{}' \delta_i r_i] = \sum_{i=1}^{m}{}' \gamma_i r_i,$$

where $\delta_i = \alpha_i \beta/[1-(1-\beta)\alpha_{m-n}]$ if $i = 1, \ldots, m-n-1$ and $\delta_i = \alpha_i(1-\beta)/[1-(1-\beta)\alpha_{m-n}]$ if $i = m-n+1, \ldots, m$.

Definitions. *Let* (R, Φ) *be a mixture set with the property* M4. *We define:*

1. *The mixture hull* S^* *of a set* $S \subset R$ *is the set of all points* $r \in R$ *expressible as* $r = \sum_{i=1}^{n}{}' \alpha_i r_i$ *for some* $n \geqq 1$, $r_i \in S$, $i = 1, \ldots, n$.

2. *A finite set* $S = \{s_1, \ldots, s_n\} \subset R$ *is dependent if there are numbers* $\alpha_1 > 0, \ldots, \alpha_n > 0$, $\sum_{i=1}^{n} \alpha_i = 1$ *and a set* $S' \subsetneq S$ *such that* $\sum_{i=1}^{n}{}' \alpha_i s_i \in (S')^*$.

3. *A not dependent finite set is independent. An arbitrary set $S \subset R$ is independent if every finite subset of S is independent.*

Lemma 7. *If (R, Φ) is a mixture set with the property M4, then to each $r \in R$ there exists a maximal independent set $S_r \subset R$ containing r.*

Proof. The one-point set $\{r\}$ is an independent set. We consider the class $\mathcal{S}$ of all independent sets containing r. $\mathcal{S}$ is partially ordered by inclusion. The union of independent sets in any linearly ordered subclass of $\mathcal{S}$ is an independent set. Therefore, every linearly ordered subclass of $\mathcal{S}$ has an upper bound in $\mathcal{S}$ and $\mathcal{S}$ contains a maximal element S_r by Zorn's lemma of the set theory.

Theorem. *A topological mixture set (R, Φ) with the property M4 is isomorphic to a convex subset of a linear space.*

Proof. 1. If S is a maximal independent subset of R, then

$$(8) \qquad (\forall r \in R)\,(\exists r_1, r_2 \in S^*)\,(\exists \alpha \in (0, 1\rangle)\,(r_1 = \Phi_\alpha(r, r_2)),$$

where S^* is the mixture hull of S.

We shall prove (8). If $r \in S$, (8) is obvious. If $r \in R - S$, then $S \cup \{r\}$ is a dependent set and there is a finite set $S_0 = \{s_1, \ldots, s_m\} \subset S$ such that $S_0 \cup \{r\}$ is a dependent set. Hence there are numbers $\alpha_0 > 0$, $\alpha_1 > 0, \ldots$, $\alpha_m > 0$, $\sum_{i=0}^{m} \alpha_i = 1$. and a set $S' \subsetneq S \cup \{r\}$ such that $\sum_{i=0}^{m}{}' \alpha_i s_i = \sum_{s_i \in S'}{}' \beta_i s_i$, where we put s_0 instead of r. We denote $r_1 = \sum_{i=1}^{m}{}' [\alpha_i/(1-\alpha_0)] s_i$, $r_2 = \sum_{s_i \in S' - \{r\}}{}' [\beta_i/(1-\beta_0)] s_i$. This is possible, since $\alpha_0 < 1$, and if $\beta_0 = 1$, then $\Phi_{\alpha_0}(r, r_1) = r$, hence, following Lemma 3, $\alpha_0 = 1$. Thus we have $\Phi_{\alpha_0}(r, r_1) = \Phi_{\beta_0}(r, r_2)$, i. e. $\Phi_{(1-\alpha_0)}(r_1, r) = \Phi_{(1-\beta_0)}(r_2, r)$, where $(1 - \beta_0) > 0$ and $(1 - \alpha_0) > 0$. According to Lemma 4 either $r_2 = \Phi_\gamma(r_1, r)$ or $r_1 = \Phi_\gamma(r_2, r)$ for some $\gamma \in (0, 1\rangle$. But if $\gamma = 1$, then $r_1 = r_2$, $r_1 \in S_0^*$, $r_2 \in (S' - \{r\})^*$, and $S' - \{r\} \subset S_0$ would be a dependent set.

2. If S is a maximal independent subset of R, r_i, $i = 1, \ldots, n$, are mutually different points from S, then

$$(9) \qquad \sum_{i=1}^{n}{}' \alpha_i r_i = \sum_{i=1}^{n}{}' \beta_i r_i \Leftrightarrow \alpha_i = \beta_i, \quad i = 1, \ldots, n.$$

We shall prove (9). If $n = 2$, then (9) coincides with Lemma 3. If $\sum_{i=1}^{n}{}' \alpha_i r_i = \sum_{i=1}^{n}{}' \beta_i r_i$ for some $n > 2$, then either $\alpha_1 = \beta_1 = 1$ or we may write in accordance with (5b) and (6)

$$\Phi_{\alpha_1}(r_1, \sum_{i=2}^{n}{}' \bar{\alpha}_i r_i) = \Phi_{\beta_1}(r_1, \sum_{i=2}^{n}{}' \bar{\beta}_i r_i), \text{ i. e.}$$

$$\Phi_{(1-\alpha_1)}(\sum_{i=2}^{n}{}' \bar{\alpha}_i r_i, r_1) = \Phi_{(1-\beta_1)}(\sum_{i=2}^{n}{}' \bar{\beta}_i r_i, r_1),$$

where $\bar{\alpha}_i = \alpha_i/(1 - \alpha_1)$, $\bar{\beta}_i = \beta_i/(1 - \beta_1)$. Suppose that $(1 - \alpha_1) \leqq (1 - \beta_1)$. From Lemma 4 we obtain $\sum_{i=2}^{n}{}' \bar{\beta}_i r_i = \Phi_\gamma(\sum_{i=2}^{n}{}' \bar{\alpha}_i r_i, r_1)$ for some $\gamma \in (0, 1\rangle$. If $\gamma < 1$, the set $\{r_1, \ldots, r_n\}$ would be dependent. Therefore $\gamma = 1$ and $\sum_{i=2}^{n}{}' \bar{\alpha}_i r_i = \sum_{i=2}^{n}{}' \bar{\beta}_i r_i$. The induction with respect to n leads to (9).

3. Let $\bar{r}$ be an arbitrary point from R, $S_{\bar{r}}$ is the maximal independent subset of R containing $\bar{r}$ and $Z = S_r - \{\bar{r}\}$. Denote by L the linear space of all finite real functions on Z assuming nonzero values only in a zero or a finite number of points from Z (with addition and multiplication by scalars defined as usual).

Denote by f_r, $r \in S_{\bar{r}}$, the function which is $f_r(z) = 1$ if $z = r$, $f_r(z) = 0$ if $z \neq r$.

If $r \in S^*_{\bar{r}}$, i.e. $r = \sum_{i=1}^{n}{}' \alpha_i r_i$ for some $r_i \in S_{\bar{r}}$, we define

$$\pi'(r) = \sum_{i=1}^{n} \alpha_i f_{r_i}. \tag{10a}$$

According to (9) π' is uniquely defined and one-to-one on $S^*_{\bar{r}}$.

If $r \in R$, then according to (8), there are $r_1, r_2 \in S^*_{\bar{r}}$ and $\alpha \in (0, 1\rangle$ such that $r_1 = \Phi_\alpha(r, r_2)$. We define

$$\pi(r) = (1/\alpha)\pi'(r_1) - [(1 - \alpha)/\alpha)]\pi'(r_2) \tag{10b}$$

If $r_1 = \Phi_\alpha(r, r_2)$, $r_3 = \Phi_\beta(r', r_4)$, $\alpha, \beta \in (0, 1\rangle$, $r_1, r_2, r_3, r_4 \in S^*_r$, then

$$\pi(r) = \pi(r') \Leftrightarrow (1/\alpha)\pi\ (r_1) + [(1 - \beta)/\beta]\pi'(r_4) =$$

$$= (1/\beta)\pi'(r_3) + [(1 - \alpha)/\alpha]\pi'(r_2) \Leftrightarrow (\beta/c)\pi'(r_1) + (1 - \beta/c)\pi'(r_4) =$$

$$(\alpha/c)\pi'(r_3) + (1 - \alpha/c)\pi'(r_2) \Leftrightarrow \Phi_{\beta/c}(r_1, r_4) = \Phi_{\alpha/c}(r_3, r_2), \tag{11}$$

where $c = \alpha + \beta - \alpha\beta$. Substituting the expression for r_1, r_3 into (11), and using M4, we obtain

$$\pi(r) = \pi(r') \Leftrightarrow \Phi_{\alpha\beta/c}(r, r_5) = \Phi_{\alpha\beta/c}(r', r_5),$$

where $r_5 = \Phi_{(1-\alpha)\beta/(\alpha+\beta-2\alpha\beta)}(r_2, r_4) \in S^*_{\bar{r}}$.

Therefore, using Lemma 4

$$\pi(r) = \pi(r') \Leftrightarrow r = r'. \tag{12}$$

From (12) it follows that the function $\pi(r) \in L$ is not dependent on the special choice of α. r_1, r_2 and π is one-to-one on R. Further, if $r \in S_r^*$, we take $r_1 = r_2 - r$, $\alpha = 1$ and $\pi(r) = \pi'(r_1)$, i.e. π and π' coincide on S_r^*.

4. If $r_1, r_2 \in S_r^*$, then, according to (10) and Lemma 6

$$\pi[\Phi_\alpha(r_1, r_2)] = \alpha\pi(r_1) + (1 - \alpha)\pi(r_2). \tag{13}$$

If r, $r' \in R$ are such that they are not both from S_r^*, we proceed as follows: Take $r_1 = \Phi_\alpha(r, r_2)$, $r_3 = \Phi_\beta(r', r_4)$, $r_1, r_2, r_3, r_4 \in S_r^*$, $0 < \beta \leqq \alpha$, $\beta < 1$ (if $1 = \beta \leq \alpha$, then r, $r' \in S_r^*$). Let us put

$$\gamma = (\alpha - \beta)/(\alpha(1 - \beta)) \in \langle 0, 1\rangle \quad \text{and} \quad \bar{r}_2 = \Phi_\gamma(r_1, r_2) \in S_r^*.$$

From M3 and (2) we obtain

$$\Phi_\beta(r, \bar{r}_2) = \Phi_\beta[r, \Phi_\gamma(\Phi_\alpha(r, r_2), r_2)] = \Phi_{\beta+(1-\beta)\alpha\gamma}(r, r_2) = \Phi_\alpha(r, r_2) = r_1.$$

Hence, for an arbitrary $\delta \in \langle 0, 1\rangle$

$$\Phi_\delta(r_1, r_3) = \Phi_\delta[\Phi_\beta(r, \bar{r}_2), \Phi_\beta(r', r_4)] = \Phi_\beta[\Phi_\delta(r, r'), \Phi_\delta(\bar{r}_2, r_4)],$$

where the last equality is obtained using Lemma 5 and Lemma 6, if we write $\sum'$ instead of Φ. Therefore, according to (10) and (13),

$$\pi[\Phi_\delta(r, r')] = (1/\beta)\pi[\Phi_\delta(r_1, r_3)] - ((1 - \beta)/\beta)\pi[\Phi_\delta(\bar{r}_2, r_4)] =$$
$$= \delta\pi(r) + (1 - \delta)\pi(r').$$

Thus $\pi(R)$ is a convex subset of L and R and $\pi(R)$ are isomorphic.

Note. Under the assumptions of the theorem, $\pi(R)$ is a topological mixture set in the topology induced by the topology of R, and π is thus a homeomorphism. This topology can be extended to the whole of L so that L becomes a topological mixture set. L will be a topological linear space only if αf will be continuous on $(-\infty, \infty) \times L$.

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