Matematický časopis

Sidney A. Morris

Free Compact Abelian Groups

Matematický časopis, Vol. 22 (1972), No. 2, 141--147

Persistent URL: http://dml.cz/dmlcz/126311

Matematický časopis 22 (1972), No. 2

FREE COMPACT ABELIAN GROUPS

SIDNEY A. MORRIS, Sydney, Australia

§1. INTRODUCTION

In this paper we give a construction of free compact abelian groups. The method is similar to that used by Gelbaum [1] to prove the existence of free topological groups. However, in Gelbaum's case the group obtained may have to be retopologized whilst this is not necessary in our case.

We show that the subgroup G of the free compact abelian group on a topological space X generated algebraically by X is the free abelian group on X. From this we deduce the existence of free abelian topological groups.

The theory of varieties of topological groups was introduced and developed in [5], [6], [7] and [8]. We show here that the group G mentioned above is the free topological group of the variety generated by the circle group. Consequently G is not a free abelian topological group.

It was shown in [4] that $C(X)$, as a semigroup, characterizes X within the class of compact Hausdorff spaces. We investigate the following question: does there exist an abelian topological group H such that the group of continuous mappings of X into H characterizes X within the class of compact Hausdorff spaces? We show that the answer is in the negative.

By the term "group,, we will mean "abelian,, group. We will denote the multiplicative group of complex numbers of modulus one (the circle group) with the usual topology by T. The cartesian product group of a set $\{G_\gamma : \gamma \in \Gamma\}$ of topological groups with the usual product topology will be denoted by $\prod_{\gamma \in \Gamma} G_\gamma$. Finally note that we write all groups multiplicatively.

§2. PRELIMINARIES

Definition. *A non-empty class $\underline{\underline{V}}$ of topological groups is said to be a variety of topological groups if it has the properties:*

(a) *if G is a subgroup of the product group $\prod_{\gamma \in \Gamma} G_\gamma$, where Γ is any index set and G_γ is in $\underline{\underline{V}}$ for $\gamma \in \Gamma$, then G is in $\underline{\underline{V}}$,*

(b) *if H is a quotient group of any G in $\underline{\underline{V}}$, then H is in $\underline{\underline{V}}$.*

Clearly a variety of topological groups determines a variety of groups [9]; the latter is simply the class of groups which with some topology appears in the former. (This is indeed a variety of groups by 15.51 of [9].)

Definition. *Let G be any topological group and $\underline{\underline{V}}(G)$ be the intersection of all varieties of topological groups containing G. Then $\underline{\underline{V}}(G)$ is said to be the topological variety generated by G.* (Clearly this is indeed a variety of topological groups.)

Definition. *Let F be in the variety $\underline{\underline{V}}$ of topological groups. Then F is said to be a free group of $\underline{\underline{V}}$ on the space X, denoted by $F(X, \underline{\underline{V}})$, if it has the properties*

(a) *X is a subspace of F,*

(b) *X generates F algebraically,*

(c) *for any continuous mapping φ of X into any H in $\underline{\underline{V}}$, there exists a continuous homomorphism Φ of F into H such that $\Phi = \varphi$ on X.*

In the particular case that $\underline{\underline{V}}$ is the variety of all topological groups, $F(X, \underline{\underline{V}})$ is called the free topological group on X.

Definition. *Let X be any Hausdorff topological space. Then the compact Hausdorff group F is said to be a free compact group on X if there exists a continuous mapping u of X into F such that,*

(i) *the subgroup of F generated algebraically by $u(X)$ is dense in $F(X)$, and*

(ii) *if φ is any continuous mapping of X into any compact group G, there exists a continuous homomorphism Φ of F into G such that $\Phi u = \varphi$.*

For a given topological space X, the uniqueness of the free topological and free compact groups on X can be deduced from the proof of Theorem 8.9 of [3].

Lemma 2.1. *Let G be a locally compact Hausdorff group and H its dual group. If A is a subgroup of H which separates points of G, then A is dense in H.* (c f.) 23.20 of [3])

Proof. Suppose A is not dense in H. Let B be the closure of A. Then H/B is a non-trivial locally compact Hausdorff group. By §1.5.2 of [10], there exists a non-trivial continuous homomorphism Φ of H/B into T.

Define the mapping ξ of H into T by $\xi(\gamma) = \Phi(B\gamma)$ for all γ in H. Then ξ is a continuous homomorphism of H into T which is not identically one but is one on B. By the Pontryagin Duality theorem ([10]) there exists an $x \neq 1$ such that $\xi(\gamma) = \gamma(x)$ for all γ in H. Then $\gamma(x) = 1$ for all γ in A. This implies $x = 1$, since A separates points of G. This is a contradiction and thus A is dense in H.

Let X be a Hausdorff topological space and $\Gamma(X)$ the group of all continuous mappings of X into T where the product of elements f and g of $\Gamma(X)$ is defined by $(fg)(x) = f(x)g(x)$ for all x in X. Put the discrete topology on $\Gamma(X)$ and let $F(X)$ be its dual group. We will show that $F(X)$ is the free compact group of X.

Lemma 3.1. *The mapping u of X into $F(X)$ given by $u(x) = \gamma_x$, $x \in X$, where $\gamma_x(f) = f(x)$ for all f in $\Gamma(X)$ is continuous. Further u is a homeomorphism of X onto $u(X)$ if and only if X is a completely regular space.*

Proof. Clearly, by Theorem 1.2.6 of [10], u is continuous. Obviously if u is a homeomorphism of X onto $u(X)$, then X is completely regular.

Let X be completely regular. Then for any pair of distinct points x and y in X, there is an f in $\Gamma(X)$ such that $f(x) \neq f(y)$. Thus u is one to one. Let O be any open set in X and γ_a be any point in $u(O)$. Since X is completely regular, there is a g in $\Gamma(X)$ such that $g(a) \neq -1$ and $\{x : g(x) \neq -1\} \subseteq O$. But $\{\gamma : \gamma \in F(X), \gamma(g) \neq -1\}$ is open in $F(X)$. Thus $\{\gamma_x : \gamma_x(g) \neq -1\}$ is an open neighbourhood of γ_a in $u(X)$, and is contained in $u(O)$. Hence u is a homeomorphism of X onto $u(X)$.

Lemma 3.2. *Let φ be a continuous mapping of a Hausdorff space X_1 into a Hausdorff space X_2. Then there exists a continuous homomorphism Φ of $F(X_1)$ into $F(X_2)$ such that the diagram below commutes.*

$$\begin{array}{ccc} F(X_1) & \xrightarrow{\Phi} & F(X_2) \\ \uparrow{\scriptstyle u_1} & & \uparrow{\scriptstyle u_2} \\ X_1 & \xrightarrow[\varphi]{} & X_2 \end{array}$$

Proof. Define $\Phi : F(X_1) \to F(X_2)$ by $\Phi(\gamma) = \xi$, $\gamma \in F(X_1)$, where $\xi(f) = \gamma(f\varphi)$ for all f in $\Gamma(X_2)$. Clearly ξ is in $F(X_2)$ and it can easily be verified that Φ is a continuous homomorphism and that the diagram commutes.

Lemma 3.3. *If X is a compact Hausdorff group, then there exists an open continuous homomorphism δ of $F(X)$ onto X such that $\delta(u(x)) = x$ for all x in X.*

Proof. Let Γ be the dual group of X. Then Γ is a topological subgroup of $\Gamma(X)$. By Lemma 24.5 of [3], the map Θ of $F(X)$ onto the dual group H of Γ, defined by $\Theta(\gamma) = \gamma \mid \Gamma$, $\gamma \in F(X)$, is an open continuous homomorphism. By the Pontryagin duality theorem, the mapping α, defined by $\alpha(x) = \gamma_x \mid H$ for all x in X, is a topological isomorphism of X onto H.

Define $\delta : F(X) \to X$ by $\delta(\gamma) = \alpha^{-1}(\Theta(\gamma))$, $\gamma \in F(X)$. Clearly δ is an open continuous homomorphism of $F(X)$ onto X. Let $\delta(\gamma_x) = y$. Then $\alpha^{-1}(\Theta(\gamma_x)) = y$; that is $\Theta(\gamma_x)f = f(y)$ for all f in Γ. Thus $f(x) = f(y)$ for all f in Γ. This implies $x = y$. Thus $\delta(u(x)) = x$ for all x in X.

Theorem 3.4. *For any Hausdorff space X, $F(X)$ is the free compact group on X.*

Proof. Clearly $F(X)$ is a compact group. Let A be the subgroup of $F(X)$ generated algebraically by $u(X)$. Then $u(X)$, and therefore A, separate points of $\Gamma(X)$. Thus, by Lemma 2.1, A is dense in $F(X)$.

Let ψ be any continuous mapping of X into any compact Hausdorff group X_1. By Lemma 3.2, there exists a continuous homomorphism Φ of $F(X)$ into $F(X_1)$ such that the diagram below commutes.

$$\begin{array}{ccc} F(X) & \xrightarrow{\ \Phi\ } & F(X_1) \\ u \uparrow & & \uparrow u_1 \\ X & \xrightarrow[\ \psi\ ]{} & X_1 \end{array}$$

Lemma 3.3 implies that there is a continuous homomorphism δ of $F(X_1)$ onto X_1 such that $\delta(u_1(x)) = x$ for all x in X_1. Define $\Psi : F(X) \to X_1$ by $\Psi(\gamma) = \delta(\Phi(\gamma))$ for all γ in $F(X)$. Clearly Ψ is a continuous homomorphism and $\Psi u = \psi$.

Corollary 3.5. *The Hausdorff space X is a subspace of its free compact group if and only if it is completely regular.*

Corollary 3.6. *Every compact Hausdorff group is a quotient group of its free compact group.*

Proof. In view of Theorem 3.4, this is just a restatement of Lemma 3.3

§4. CONSEQUENCES

Theorem 4.1. *The subgroup A of $F(X)$ algebraically generated by $u(X)$ is (algebraically) a free group on $u(X)$.*

Proof. Suppose $a = u(x_1)^{\varepsilon_1} \dots u(x_n)^{\varepsilon_n} = 1 \in A$, where $u(x_i) \neq u(x_j)$ for $i \neq j$ and ε_i is a non-zero integer for each i. Let O be an open set in $u(X)$ which contains $u(x_1)$ but not $u(x_i)$ for $i \neq 1$. Since $u(X)$ is completely regular, there exists a continuous map Θ of $u(X)$ into $\{z : z = e^{it}, O \leq t \leq 1\} \subset T$ such that $\Theta(y) = 1$ for all y not in O and $\Theta(u(x_1)) = e^{i}$.

Let φ Θu. Then φ is a continuous mapping of X into T. Therefore there exists a continuous homomorphism Φ of $F(X)$ into T such that Φu φ. Then $\Phi(a)$ $e^{i\varepsilon_1} \neq 1$. This is clearly a contradiction and the theorem is proved.

Corollary 4.2. *Let X be a completely regular Hausdorff space. Then the free topological group on X exists and is Hausdorff.*

Proof. By comments in §2 of [2] it is sufficient to prove that there exists some Hausdorff group topology on the free group on X, which indices the given topology on X.

Lemma 3.1 implies X is a subspace of $F(X)$ and Theorem 4.1 implies that the subgroup of $F(X)$ generated algebraically by X is the free group on X. The proof is complete.

Remark 4.3. We are led to ask: If X is a completely regular Hausdorff space, is the subgroup A of $F(X)$ algebraically generated by X (actually $u(X)$) the free topological group on X? If this is not true in general is it true for some X? We will show in Theorem 4.4 that the answer to each of the questions is in the negative.

Theorem 4.4. *If X is a completely regular Hausdorff space, then the subgroup A of $F(X)$, algebraically generated by X, is the free group $F(X, \underline{\underline{V}}(T))$ on X of the topological variety $\underline{V}(T)$ generated by T. Consequently A is not the free topological group on X.*

Proof. Theorem 5.4 of [7] shows that $F(X)$ is in $\underline{V}(T)$. Therefore A is in $V(T)$. Thus by Theorem 2.6 of [5], $F(X, \underline{\underline{V}}(T))$ exists. Clearly $F(X, V((T))$ is algebraically isomorphic to A and has a finer topology than A. Consequently $F(X, \underline{V}(T))$ is Hausdorff which implies by Lemma 5.3 of [7] that it can be imbedded in a compact Hausdorff group H.

Let φ be the identity mapping: $X(\subseteq F(X)) \to X(\subseteq F[X, \underline{\underline{V}}(T)])$. Then φ is a continuous mapping of X into H. Theorem 3.4 implies that there exists a continuous homomorphism Φ of $F(X)$ into H such that Φ X φ. The map $\Phi \mid A$ is a continuous algebraic isomorphism of A onto $F(X, V(T))$. Therefore A has a finer topology than $F(X, \underline{\underline{V}}(T))$. Hence A is topologically isomorphic to $F(X, \underline{V}(T))$.

The final remark in the theorem now follows immediately from Theorem 7.28 of [8].

Corollary 4.5. *The topological variety $\underline{\underline{V}}(T)$ is a β-variety* ([6]).

Proof. This statement is equivalent to the fact that $F(X, V(T))$ exists and is Hausdorff, which is proved in Theorem 4.4.

The following theorem which appears in [1] is an immediate consequence of Theorem 3.4.

Theorem 4.6. *Let X be a completely regular Hausdorff space. Then $F(X)$ is the Bohr compactification of the free topological group on X.*

We point out that Theorem 3.4 could have been proved using Theorem 4.6. Whilst this proof would have been shorter in the case that X is completely regular, the proof in the more general case would not.

§5. CHARACTERIZATION OF TOPOLOGICAL SPACES

In this section all topological spaces considered will be Hausdorff. A. N. Milgram [4] showed that if X and Y are compact spaces such that $C(X)$ and $C(Y)$ are isomorphic semigroups then X and Y are homeomorphic. We are therefore led to ask the question: does $\Gamma(X)$ characterize X within the class of compact spaces? The following theorem, together with results in [2] shows that the answer is in the negative.

Theorem 5.1. Let X and Y be topological spaces with topologically isomorphic free topological groups. Then their free compact groups are topologically isomorphic. Further, $\Gamma(X)$ and $\Gamma(Y)$ are isomorphic.

Proof. This is an immediate consequence of Theorem 4.6 and the Pontryagin duality theorem.

We point out that it is shown in [2] that there do exist non-homeomorphic compact spaces with topologically isomorphic free topological groups.

Let G be any topological group and X any topological space. Define $\Gamma(G, X)$ to be the group of all continuous mappings of X into G with the obvious group structure.

We are led to ask the question: does there exist a topological group G such that $\Gamma(G, X)$ characterizes X within the class of compact spaces? The following theorem shows that the answer is again in the negative.

Theorem 5.2. *Let X and Y be topological spaces with topologically isomorphic free topological groups. Then for any topological group G, $\Gamma(G, X)$ is isomorphic to $\Gamma(G, Y)$.*

Proof. Let F be the free topological group on X. Then for any continuous mapping φ of X into G, there is a unique continuous homomorphism Φ of F into G such that $\Phi u = \varphi$. Thus there is a one-one correspondence $\varphi \leftrightarrow \Phi$ between continuous mappings of X into G and continuous homomorphisms of F into G. In fact this gives an algebraic isomorphism between $\Gamma(X)$ and the group of continuous homomorphisms of F into G. Thus F determines $\Gamma(X)$. From this the result immediately follows.

REFERENCES

[1] GELBAUM, B. R.: Free topological groups. Proc. Amer. Math. Soc. *12*, 1961, 737–743.

[2] GRAEV, M. I.: Free topological groups. Izvestiya Akad. Nauk. SSSR Ser. Mat. *12*, 1948, 279–324 Russian.
English transl., Amer. Math. Soc. Translation No. *35*, 61 pp., 1951. Reprint Amer. Math. Soc. Transl. (1) *8*, 1962, 305–364.

[3] HEWITT, E.–ROSS, K. A.: Abstract harmonic analysis. Academic Press, New York, 1963.

[4] MILGRAM, A. N.: Multiplicative semigroups of continuous functions. Duke Math. J., *16*, 1949, 373–383.

[5] MORRIS, S. A.: Varieties of topological groups. Bull. Austral. Math. Soc., *1*, 1969, 145–160.

[6] MORRIS, S. A.: Varieties of topological groups II. Bull. Austral. Math. Soc., *2*, 1970, 1–13.

[7] MORRIS, S. A.: Varieties of topological groups III. Bull. Austral. Math. Soc. 2, 1970, 165–178.

[8] MORRIS, S. A.: Varieties of topological groups. Ph. D. Thesis, The Flinders University of South Australia, February 1970.

[9] NEUMANN, H.: Varieties of groups. Ergebnisse der Mathematik, und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1967.

[10] RUDIN W.: Fourier analysis on groups. Interscience Publishers, New York and London, 1960.

Received March 12, 1970

The Flinders University of South Australia
Bedford Park, South Australia
and University of New South Wales, Kensington, N.S.W. Australia